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Stochastic PDEs beyond standard monotonicity: Well posedness and regularity of solutions

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Declaration

I declare that this report was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualification.

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Abstract

Nonlinear stochastic partial differential equations (SPDEs) are used to model wide variety of phenomena in physics, engineering, finance and economics. In many such models the equations exhibit super-linear growth. In general, equations with super-linear growth are ill-posed. However if the growth satisfies some monotonicity-like conditions, then well-posedness can be shown. This thesis focuses on SPDEs that satisfy monotonicity-like conditions and consists of two main parts.

In part one, we have generalised the results using local-monotonicity condition by establishing the existence and uniqueness of solution to nonlinear stochastic partial differential equations (SPDEs) when the coefficients satisfy local monotonicity condition. This is done by identifying appropriate coercivity condition which helps in obtaining the desired higher order moment estimates without explicitly restricting the growth of the operators acting on the solution in the stochastic integral terms. As a result, we can solve various semilinear and quasilinear stochastic partial differential equations with locally monotone operators, where derivatives may appear in the operator acting on the solution under the stochastic integral term. Examples of such equations are stochastic reaction-diffusion equations, stochastic Burger equations and stochastic p-Laplace equations where the diffusion operator need not necessarily be Lipschitz continuous. Further, the operator appearing in bounded variation term is allowed to be the sum of finitely many operators, each having different analytic and growth properties. As an application, well-posedness of the stochastic anisotropic p-Laplace equation driven by Lévy noise has been shown.

In second part of this thesis, new regularity results for solution to semilinear SPDEs on bounded domains are obtained. The semilinear term is continuous, monotone except around the origin and is allowed to have polynomial growth of arbitrary high order. Typical examples are the stochastic Allen-Cahn and Ginzburg-Landau equations. This is done by obtaining some L^p-estimates which are subsequently employed in obtaining higher regularity of solutions. This is motivated by ongoing work to obtain rate of convergence estimates for numerical approximations to such equations.

Key words. Stochastic Partial Differential Equations, Local Monotonicity, Coercivity, Lévy Noise, Anisotropic p-Laplace Equation, Regularity, Weighted Sobolev Spaces.
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Lay Summary

This thesis concerns stochastic partial differential equations (SPDEs). Partial differential equation (PDE) is a mathematical equation that relates some function of several variables with its derivatives. They model a wide variety of phenomena in physics, engineering, biology, finance and economics (sound, heat transfer, diffusion, electromagnetism, fluid dynamics, elasticity, filtering, population dynamics, option prices, economic equilibria). Many models include random phenomena or uncertainty and such uncertainty can be modelled by adding stochastic terms (e.g. white noise) to the partial differential equation.

An important question is whether such an equation is well-posed: in other words, does a solution exist? Is it unique? And does it vary continuously if we change the inputs to the equation (initial or boundary conditions). If an equation is ill posed, then extra care is needed to use this as a model of nature and well-posed problems are thus more convenient to work with.

SPDEs have been the subject of intensive study over the last half century and much has been discovered. In this thesis we contribute two types of results. The first one states that for nonlinear SPDEs where the non-linearity satisfies appropriate but very general monotonicity-like condition then the equation is well-posed. Such monotonicity assumptions have been used before but here the contribution is that we allow a very general one and also show that in some sense this cannot be weakened further. The second result concerns regularity of the solutions. By regularity we mean whether the solution function varies smoothly or roughly and exactly how smooth or how rough it is. While this is interesting from modelling perspective in its own right it has also important implication for numerical approximations of solutions: the smoother the solution the better numerical approximation can be devised. This is very useful since in real life applications, it is impossible to write the solution to the SPDE in an explicit form and one needs to rely on the numerical approximation.

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To my soul mate Chaman,
Maa, Paa & sisters

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Chapter 1

Introduction

Stochastic evolution equations (SEEs) or stochastic partial differential equations (SPDEs) have been intensively investigated in the literature, motivated by wide ranging applications. For equations with non-linear drift or diffusion operators, the introduction of variational method and the theory of monotone operators lead to an interesting area of research.

This thesis is divided in two parts. Part one consists of Chapters 2 and 3, where we have presented the existence and uniqueness results for a large class of nonlinear SPDEs driven by Wiener noise and Lévy noise respectively. To be precise we have generalised many previous works [3, 28, 30, 39] by identifying an appropriate coercivity condition, which allow us to solve SPDEs when the coefficients satisfy local monotonicity condition, without explicitly restricting the growth of the operators acting on the solution in the stochastic integral term. Chapter 4, which form the second part of this thesis, contains some new results about regularity of solutions to semilinear SPDEs on bounded domains.

1.1 Literature review and brief summary of results

Let $T > 0$ be given and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis, i.e. $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space equipped with a right continuous filtration $(\mathscr{F}_t)_{t\in[0,T]}$ such that $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets. Let $W := (W_t)_{t\in[0,T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$, i.e. the coordinate processes $(W^j_t)_{t\in[0,T]}, j \in \mathbb{N}$ are independent $\mathscr{F}_t$-adapted Wiener processes and $W^j_t - W^j_s$ is independent of $\mathscr{F}_s$ for $s \leq t$. Further assume that $(H, (\cdot,\cdot), |\cdot|_H)$ is a separable Hilbert space. Let $(V, |\cdot|_V)$ be a separable, reflexive Banach space embedded continuously and densely in H with $(V^*, |\cdot|_{V^*})$ denoting its dual and $\langle\cdot,\cdot\rangle$ the duality pairing between V and V^*. Identifying H with H^* using the Riesz representation and the inner product in H, one obtains the Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where $\hookrightarrow$ denotes continuous and dense embedding.

Consider the stochastic evolution equation

$$u_t = u_0 + \int_0^t A_s(u_s)ds + \sum_{j=1}^{\infty} \int_0^t B^j_s(u_s)dW^j_s, \quad t \in [0,T], \tag{1.1}$$

where the initial condition u_0 is an H-valued $\mathscr{F}_0$-measurable random variable. Moreover A and $B^j, j \in \mathbb{N}$, are progressively measurable non-linear operators mapping $[0,T] \times \Omega \times V$ into V^* and H respectively. Here, one should note that the formulation of (1.1) is equivalent to considering the analogous equation driven by a cylindrical Wiener process, see Appendix A.

The nonlinear SEE (1.1) has been initially studied in Pardoux [39] and Krylov and Rozovskii [28], where a priori estimates are proved, giving the second moment estimates under what are now classical monotonicity, coercivity and growth assumptions. The estimates so obtained allow the authors to obtain existence and uniqueness of solutions to (1.1). One of the key results in [28] is the theorem about Itô's formula for the square of the norm of a continuous semimartingale in a Gelfand triple obtained separately from the related SEE. This theorem provides the continuity of the solution in the pivot space of the Gelfand triple and is the key to obtain the a priori estimates and prove the existence and uniqueness of the solution. These, now classical results, have been generalized in a number of directions. Of those one notes the inclusion of general càdlàg semimartingales as the driving process in stochastic integral, see Gyöngy and Krylov [13] and Gyöngy [11]. Closely related to the results in this thesis is the work by Liu and Röckner [30] (or [32]). They extended the framework of Krylov and Rozovskii [28] to stochastic evolution equations when the operators are only locally monotone and the operator A, which is the operator acting in the bounded variation term, satisfies a less restrictive growth condition. To obtain a generalization in this direction Liu and Röckner [30] need higher order moment estimates and to obtain them they place a restrictive assumption on the growth of the operator B (i.e. (2.5)), which is the operator acting on the solution under the stochastic integral. As a consequence one may not have derivatives appearing in this operator. The local monotonicity and coercivity conditions are further weakened in Liu and Röckner [31] but again at the expense of having a growth restriction on the operator B. Moreover, Brzeźniak, Liu and Zhu [3] extend the results in [30] to include equations driven by Lévy noise but again with suboptimal growth restrictions on the operators appearing under the stochastic integrals (see also Remark 2.5).

In Chapter 2, we have identified appropriate coercivity assumption, which we call as p_0-stochastic coercivity, which allows us to obtain higher order moment estimates and to prove existence and uniqueness of solution to (1.1) without the need to explicitly restrict the growth of the operator B. To be exact, we prove our results without requiring the first inequality in (1.2) in [30] or equivalently in (5.2) in [32]. Examples of SPDEs which do not fit the framework of Krylov and Rozovskii [28] or Liu and Röckner [30, 32] or Brzeźniak, Liu and Zhu [3] but which fit into the setting of the chapter are given. Finally, an example is considered that, together with results from Brzeźniak and Veraar [4], shows that the coercivity assumption we have identified is the optimal one.

Models based on SPDEs driven by jump type noises have become extremely popular in recent years. Thus in Chapter 3, we have extended the results of Chapter 2 to include SEEs driven by Lévy noise. Further, the drift term is allowed to be the sum of finitely many operators each having different analytic and growth properties. An example of such equation is an stochastic anisotropic p-Laplace equation driven by Lévy noise. Such an equation in deterministic setting has been considered by Lions [29]. Similar to the results in Chapter 2, we have obtained the existence and uniqueness results for SEEs, considered in Chapter 3, under appropriate coercivity condition without the need to explicitly restrict the growth of the operators acting on the solution in stochastic integral terms, precisely the inequality (1.2) in [3].

Another area of active interest, in the theory of nonlinear SPDEs, is studying the regularity of the solutions. Results on regularity of solutions to linear SPDEs on the whole space have been obtained by Rozovskii [43]. However, regularity of solutions to SPDEs on domains with boundary is a difficult problem and one cannot expect the same regularity up to the boundary as in the interior of the domain. The following example from Krylov [24] demonstrates that on a domain with boundary, even for a very good data the solution may not have good second derivatives upto the boundary.

Example 1.1. Consider the following SPDE :

$$\begin{cases} du(t,x) = \partial_x^2 u(t,x)dt + dW(t) \quad \text{on} \quad (0,1)\times(0,1), \\ u(t,0) = u(t,1) = 0 \qquad \forall \quad t \in (0,1), \\ u(0,x) = 0 \qquad \forall \quad x \in (0,1), \end{cases}$$

where $W(t)$ is a one-dimensional Wiener process. Now, if we assume that the function $\partial_x^2 u(t,x)$ is continuous in x on $[0,1]$, bounded in (t,x) and integrable in t for each $x \in [0,1]$, then using the continuity at $x = 0$ we get

$$0 = u(t,0) = \lim_{x\to 0} u(t,x) = \int_0^t \lim_{x\to 0} \partial_x^2 u(s,x)ds + W(t)$$

which implies

$$W(t) = -\int_0^t \partial_x^2 u(s,0)ds$$

which is a contradiction since $W(t)$ does not have bounded variation.

After this observation two approaches to deal with boundaries emerge: one is to quantify the loss of regularity near the boundary using weighted Sobolev spaces. These allow oscillations and explosion of the spatial derivatives of the solution near the boundary. The other approach is to side-step the problems created by the boundary by restricting the class of equations under consideration by imposing additional restriction on the noise term near the boundary (effectively disallowing stochastic forcing near the boundary), see Flandoli [8]. Weighted Sobolev spaces have also been employed, in the context of L^p-theory for linear SPDEs, by Kim [20].

Unsurprisingly, there are fewer results for nonlinear SPDEs. Kim and Kim use the L^p-theory in [21] and [22] to obtain regularity for quasilinear SPDEs where the coefficients are uniformly bounded. Current results in Gerencsér [9] show that for a class of SPDEs, including (1.2) mentioned below, there exists some Hölder exponent such that the solution is Hölder continuous in space up to the boundary with this exponent. For interior regularity of a class of quasilinear equations associated with the "p-Laplace" operator see Breit [1]. For SPDEs with drift given by the subgradient of a quasi-convex function and with sufficiently regular noise, Gess [10] proves higher regularity and existence of (analytically) strong solutions. All the aforementioned work on regularity of nonlinear SPDEs has been done using the variational approach. For results obtained in the semigroup framework we refer the reader to the work of Jentzen and Röckner [18] and references therein.

In the second part of this thesis, we discuss the regularity of solutions to the semilinear SPDE,

$$\begin{aligned} du_t &= (L_t u_t + f_t(u_t, \nabla u_t) + f_t^0)dt + \sum_{j=1}^{\infty}(M_t^j u_t + g_t^j)dW_t^j \quad \text{on } [0,T]\times\mathscr{D}, \\ u_t &= 0 \text{ on } \partial\mathscr{D}, \quad u_0 = \phi \text{ on } \mathscr{D}. \end{aligned} \tag{1.2}$$

where,

$$L_t u := \sum_{k=1}^d \partial_k\Big(\sum_{i=1}^d a_t^{ik}\partial_i u\Big) + \sum_{i=1}^d b_t^i \partial_i u + c_t u \quad \text{and} \quad M_t^j u := \sum_{i=1}^d \sigma_t^{ij}\partial_i u + \mu_t^j u.$$

Here $\mathscr{D}$ is a bounded domain in $\mathbb{R}^d$ and W^j, as defined above, are independent Wiener processes. The coefficients a and σ are assumed to satisfy stochastic parabolicity condition (and thus our equation is non-degenerate). Moreover all the coefficients a, b, c, σ and μ are assumed to be measurable and bounded, $f = f_t(\omega, x, r, z)$ is measurable, continuous in (r,z), monotone in r except perhaps around the origin, Lipschitz continuous in z, bounded in x and of polynomial growth in r (of arbitrary order). The forcing terms f^0 and g are assumed to satisfy appropriate integrability conditions. A typical example of equation fitting in this setting is the stochastic Ginzburg–Landau equation. In this case,

$$f(r) = -|r|^{\alpha-2}r\,, \;\; \alpha \geq 2.$$

To obtain higher interior regularity we will have to impose further regularity assumptions on the coefficients. To obtain regularity up to the boundary (in weighted Sobolev spaces) we will also need to impose regularity assumptions on the domain. The assumptions will be formulated precisely in Chapter 4.

In Chapter 4, we have obtained regularity results for the solutions to the SPDE (1.2). This is motivated by ongoing work to obtain rate of convergence estimates for numerical approximations to such equations. For a semilinear equation it is natural to consider the term $f := f(u, \nabla u) + f^0$ as a free term in an appropriate linear SPDE and to use established methods and theory to obtain regularity for this linear SPDE. Due to uniqueness of solutions to (1.2), see Lemma 4.1, we then get the same regularity for the semilinear equation (1.2). However, to apply the theory of regularity of linear SPDEs, we need to show that the new free term f satisfies appropriate integrability conditions. This would typically mean at least L^2-integrability. Since the semilinear term in (1.2) is allowed to have arbitrary polynomial growth, it is clear that we need to obtain L^p-estimates for solution to (1.2) with $p \geq 2$ sufficiently large. Note that if one attempts to do this using Sobolev embedding theorem, then one immediately runs into restrictions on the combination of dimension of $\mathscr{D}$ and the growth of the semilinear term. The main novelty of our result is in allowing arbitrary dimension of $\mathscr{D}$ and growth of the semilinear term, see Theorem 4.1. This is achieved by using the monotonicity property of the semilinear term and a cutting argument to obtain the required L^p-estimate. Once these have been established, we then obtain new spatial regularity results for the SPDE (1.2). These are both interior regularity and up-to-the-boundary regularity in weighted Sobolev spaces, see Theorems 4.2 and 4.5. Finally we have a new time regularity result (in weighted space again), see Theorem 4.6. These results effectively say that under appropriate assumptions the SPDE (1.2) has two additional derivatives. It seems however that our method does not allow one to obtain arbitrarily high regularity (even for equation with smooth data and coefficients), see Remark 4.5 for explanation. Nevertheless, raising the regularity twice is enough to find the rate of convergence of various numerical approximations using the techniques from e.g. Gyöngy and Millet [15]. One such example is presented in Section 4.4.

Finally in Chapter 5, we have presented some ideas of possible work that can be done in near future.

As mentioned earlier, in Appendix A we have shown that the SEE (1.1) is equivalent to considering the analogous SEE (A.1) driven by a cylindrical Wiener process taking values in a separable Hilbert space.

1.2 Notations

Besides the notations $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ for a stochastic basis, $W := (W_t)_{t\in[0,T]}$ for an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$ and a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$, that have already been introduced in the previous section, we will use the following notations throughout this thesis.

Let $(Z, \mathscr{Z}, \nu)$ be a σ-finite measure space and $N(dt, dz)$ be a Poisson random measure defined on $(Z, \mathscr{Z}, \nu)$ with intensity ν. The compensated Poisson random measure is denoted by $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$.

For a fixed $T > 0$, we denote the predictable σ-algebra on $[0, T] \times \Omega$ by $\mathscr{P}$. The indicator function of a set A is denoted by $\mathbf{1}_A$. For two real numbers a and b, their minimum is denoted by $a \wedge b$.

The set of natural numbers is denoted by $\mathbb{N}$ and for each $d \in \mathbb{N}$, we denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. For a topological space X, we denote the Borel σ-algebra on X by $\mathscr{B}(X)$. In general, if X is a normed linear space then we will use $|\cdot|_X$ to denote the norm in this space. There is an exception: if $x \in \mathbb{R}^d$, then $|x|$ denotes the Euclidean norm.

Let $(X, |\cdot|_X)$ be a Banach Space. For a given constant $p \in [1, \infty)$, $L^p(\Omega; X)$ denotes the Bochner–Lebesgue space of equivalence classes of random variables x taking values in X such that the norm,

$$|x|_{L^p(\Omega;X)} := (\mathbb{E}|x|^p_X)^{\frac{1}{p}}$$

is finite. Again, $L^p((0, T); X)$ denotes the Bochner–Lebesgue space of equivalence classes of X-valued measurable functions such that the norm,

$$|x|_{L^p((0,T);X)} := \Big(\int_0^T |x_t|^p_X \, dt \Big)^{\frac{1}{p}}$$

is finite while $L^\infty((0, T); X)$ denotes the Bochner–Lebesgue space of X-valued measurable functions which are essentially bounded, i.e.

$$|x|_{L^\infty((0,T);X)} := \operatorname*{ess\,sup}_{t\in(0,T)} |x_t|_X < \infty \, .$$

Finally, $L^p((0,T)\times\Omega;X)$ denotes the Bochner–Lebesgue space of equivalence classes of X-valued stochastic processes which are progressively measurable and the norm,

$$|x|_{L^p((0,T)\times\Omega;X)} := \Big(\mathbb{E}\int_0^T |x_t|_X^p\,dt\Big)^{\frac{1}{p}} < \infty\,.$$

Further, we denote by $C([0,T];X)$ the space of X-valued continuous functions on $[0,T]$ and by $D([0,T];X)$ the space of X-valued càdlàg (continuous from the right and limit from the left) functions on $[0,T]$.

We use the notation $\ell^2(X)$ to denote the space of all square-summable X-valued sequences. However, the space of real valued sequences is denoted by ℓ^2.

If x_n is a sequence in X converging to x strongly (i.e. in norm topology), then we denote this fact by $x_n \to x$ whereas we use the notation $x_n \rightharpoonup x$ if x_n converges to x weakly (i.e. in the weak topology).

In the special case $X=\mathbb{R}$, we will use the following notations.

Let $\mathscr{D}\subseteq\mathbb{R}^d$ be an open bounded domain with smooth boundary unless mentioned otherwise. Then for any $p\geq 1$, $L^p(\mathscr{D})$ is the Lebesgue space of equivalence classes of real valued measurable functions u defined on $\mathscr{D}$ such that the norm,

$$|u|_{L^p} := \Big(\int_{\mathscr{D}} |u(x)|^p dx\Big)^{\frac{1}{p}}$$

is finite. For $i\in\{1,2,\ldots,d\}$, let D_i be the distributional derivative along the i-th coordinate in $\mathbb{R}^d$ and $\nabla := (D_1, D_2,\ldots,D_d)$ be the gradient. Further for a multi-index $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_d)$ of non-negative integers, its order is denoted by $|\gamma| := \gamma_1+\gamma_2+\ldots+\gamma_d$ and the operator $D^\gamma := D_1^{\gamma_1}D_2^{\gamma_2}\ldots D_d^{\gamma_d}$. We denote by $W^{1,p}(\mathscr{D})$ the Sobolev space of real valued functions u, defined on $\mathscr{D}$, such that the norm

$$|u|_{1,p} := \Big(\int_{\mathscr{D}} \big(|u(x)|^p + |\nabla u(x)|^p\big)\,dx\Big)^{\frac{1}{p}} < \infty\,.$$

Let $C_0^\infty(\mathscr{D})$ be the space of smooth functions with compact support in $\mathscr{D}$. Then, the closure of $C_0^\infty(\mathscr{D})$ in $W^{1,p}(\mathscr{D})$ with respect to the norm $|\cdot|_{1,p}$ is denoted by $W_0^{1,p}(\mathscr{D})$. Friederichs' inequality (see, e.g. Theorem 1.32 in [42]) implies that the norm

$$|u|_{W_0^{1,p}} := \Big(\int_{\mathscr{D}} |\nabla u(x)|^p\,dx\Big)^{\frac{1}{p}}$$

is equivalent to $|u|_{1,p}$ and this equivalent norm $|u|_{W_0^{1,p}}$ will be used throughout this thesis. Note that for Lebesgue and Sobolev spaces over the entire domain $\mathscr{D}$ we have omitted the dependence on $\mathscr{D}$ in the notation for norm, i.e., for $u\in L^p(\mathscr{D})$, we write $|u|_{L^p}$ for $|u|_{L^p(\mathscr{D})}$.

Furthermore, by $W^{-1,p}(\mathscr{D})$ we denote the dual of $W_0^{1,p}(\mathscr{D})$ and $|\cdot|_{W^{-1,p}}$ is used to denote the norm on this dual space. It is well known that

$$W_0^{1,p}(\mathscr{D}) \hookrightarrow L^2(\mathscr{D}) \equiv (L^2(\mathscr{D}))^* \hookrightarrow W^{-1,p}(\mathscr{D}),$$

is a Gelfand triple. Next we define the Laplacian operator, $\Delta : W_0^{1,2}(\mathscr{D}) \to W^{-1,2}(\mathscr{D})$ by

$$\langle \Delta u, v\rangle := -\int_{\mathscr{D}} \nabla u(x)\nabla v(x)dx\,, \qquad \forall v\in W_0^{1,2}(\mathscr{D})\,. \tag{1.3}$$

Clearly, using Hölder's inequality and the definition of dual norm, we have

$$|\Delta u|_{W^{-1,2}} \leq |u|_{W_0^{1,2}} \tag{1.4}$$

and so the operator 'Δ' is linear and bounded. Throughout this thesis, C is a generic constant that may change from line to line.

1.3 Some useful results

In this section, we briefly present some key results which are relevant for this thesis. Due to brevity of space, they are stated without proof except where the proof is required. However the details can be found in [2, 19, 34, 42]. First we state some important inequalities which will be used throughout this thesis.

Lemma 1.1 (Young's Inequality)**.** *Let a and b be non-negative real numbers. Then for any $\epsilon > 0$,*

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{q}{p}}} b^q$$

for any $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.2 (Hölder's Inequality)**.** *Consider a measure space $(X, \mathcal{M}, \mu)$ and let $f, g : X \to [0, \infty]$ be measurable. Then*

$$\int_X fg d\mu \leq \Big(\int_X f^p d\mu \Big)^{\frac{1}{p}} \Big(\int_X g^q d\mu \Big)^{\frac{1}{q}}$$

for any $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.3 (Gronwall's Inequality)**.** *Let $T > 0$ and $\alpha \geq 0$ be fixed constants. Let $u(\cdot)$ be a bounded non-negative Borel measurable function defined on $[0, T]$ and $v(\cdot)$ be a non-negative integrable function defined on $[0, T]$. If*

$$u(t) \leq \alpha + \int_0^t v(r) u(r) dr$$

for all $0 \leq t \leq T$, then

$$u(t) \leq \alpha \exp \Big(\int_0^t v(r) dr \Big)$$

for all $0 \leq t \leq T$.

Lemma 1.4 (Burkholder–Davis–Gundy Inequality)**.** *Let $M := (M_t)_{t \in [0,T]}$ be a continuous local martingale defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, \mathbb{P})$ issuing from $M_0 = 0$ and $[M]$ be its quadratic variation process. Then for any $0 < p < \infty$, there exists positive constants c_p and C_p such that*

$$c_p \mathbb{E}[M]_t^{\frac{p}{2}} \leq \mathbb{E} \sup_{0 \leq s \leq t} |M|^p \leq C_p \mathbb{E}[M]_t^{\frac{p}{2}}$$

for all $t \geq 0$.

The proof of Lemma 1.5 can be found in [35].

Lemma 1.5. *Let $r \geq 2$ and $T > 0$. There exists a constant K, depending only on r, such that for every real-valued, $\mathscr{P} \times \mathscr{Z}$-measurable function γ satisfying*

$$\int_0^T \int_Z |\gamma_t(z)|^2 \nu(dz) dt < \infty$$

almost surely, the following estimate holds,

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \Big| \int_0^t \int_Z \gamma_s(z) \tilde{N}(ds, dz) \Big|^r \leq & K \mathbb{E} \Big(\int_0^T \int_Z |\gamma_t(z)|^2 \nu(dz) dt \Big)^{\frac{r}{2}} \\ & + K \mathbb{E} \int_0^T \int_Z |\gamma_t(z)|^r \nu(dz) dt \,. \end{aligned} \tag{1.5}$$

It is known that if $1 \leq r \leq 2$, then the second term in (1.5) can be dropped.

In the following three inequalities we assume that $\mathscr{D} \subseteq \mathbb{R}^d$ is an open bounded domain with Lipschitz boundary. The first inequality follows easily using Hölder's inequality.

Lemma 1.6 (Interpolation inequality). *For, $1 \le p_1, p_2 \le \infty$, $\lambda \in [0,1]$ and p satisfying,*

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2},$$

we have,

$$|u|_{L^p(\mathscr{D})} \le |u|^{\lambda}_{L^{p_1}(\mathscr{D})} |u|^{1-\lambda}_{L^{p_2}(\mathscr{D})}.$$

Lemma 1.7 (Sobolev Embedding). *For $1 \le p < \infty$, the continuous embedding*

$$W^{1,p}(\mathscr{D}) \subset L^{p^*}(\mathscr{D})$$

holds, where

$$p^* := \begin{cases} \frac{dp}{d-p} & \textit{if } p < d \\ \textit{an arbitrary large real number} & \textit{if } p = d \\ +\infty & \textit{if } p > d, \end{cases}$$

Lemma 1.8 (Gagliardo–Nirenberg inequality). *Let $j \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$, r, q and p satisfy*

$$\frac{1}{r} = \frac{j}{d} + \lambda\Big(\frac{1}{p} - \frac{k}{d}\Big) + (1-\lambda)\frac{1}{q}, \qquad \frac{j}{k} \le \lambda \le 1, \qquad 0 \le j \le k-1,$$

then there exists a constant C such that

$$|D^j u|_{L^r(\mathscr{D})} \le C |D^k u|^{\lambda}_{L^p(\mathscr{D})} |u|^{1-\lambda}_{L^q(\mathscr{D})}.$$

The following are some consequence of Gagliardo–Nirenberg inequality. If $d = 1$, then there exists a constant C such that

$$|u|_{L^4} \le C |u|^{\frac{3}{4}}_{L^2} |u|^{\frac{1}{4}}_{W_0^{1,2}} \le C |u|^{\frac{1}{2}}_{L^2} |u|^{\frac{1}{2}}_{W_0^{1,2}}. \tag{1.6}$$

Further, if $d = 2$, then there exists a constant C such that

$$|u|_{L^4} \le C |u|^{\frac{1}{2}}_{L^2} |u|^{\frac{1}{2}}_{W_0^{1,2}}. \tag{1.7}$$

The following lemma (see, e.g. Yor [41, Chapter IV, Proposition 4.7] or Gyöngy and Krylov [14, Lemma 3.2 and Remark 3.3] for its continuous analogue) is needed to obtain some a priori estimates.

Lemma 1.9. *Let Y be a positive, adapted, right continuous process and A be a continuous increasing process. If*

$$\mathbb{E}[Y_\tau | \mathscr{F}_0] \le \mathbb{E}[A_\tau | \mathscr{F}_0]$$

for any bounded stopping time τ, then for any $r \in (0,1)$,

$$\mathbb{E} \sup_{t \ge 0} Y_t^r \le \frac{2-r}{1-r} \mathbb{E} \sup_{t \ge 0} A_t^r.$$

We end this section with the following lemma which allows us to obtain weakly-star convergent subsequences, under appropriate assumptions. The result is not new. However we could not find its proof in the literature.

Lemma 1.10. *Let X be a separable Banach space with dual X^* and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. If (S, Σ, μ) is a measure space with $\mu(S) < \infty$, and $(u_n)_{n \in \mathbb{N}}$ is a sequence satisfying*

$$\sup_n \int_S |u_n|^p_{X^*} d\mu < \infty \tag{1.8}$$

for some $1 < p < \infty$*, then there exists a subsequence* (n_k) *and* $u \in L^p(S, X^*)$ *such that* (u_{n_k}) *converges weakly-star to* u *as* $n_k \to \infty$*, i.e.,*

$$\int_S \langle u_{n_k}, \varphi \rangle d\mu \to \int_S \langle u, \varphi \rangle d\mu \qquad \forall \varphi \in L^{\frac{p}{p-1}}(S, X).$$

Proof. Let $(\phi_i)_{i \in \mathbb{N}}$ be a dense subset in X. Then, it is sufficient to show that

$$\int_S \langle u_{n_k}, \phi_i \rangle \psi d\mu \to \int_S \langle u, \phi_i \rangle \psi d\mu \qquad \forall i \in \mathbb{N}, \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R})$$

for some subsequence (n_k) and $u \in L^p(S, X^*)$. Observe that, in view of Hölder's inequality and (1.8), we have

$$\int_S |\langle u_n, \phi_i \rangle|^p d\mu \le \int_S |u_n|_{X^*}^p |\phi_i|_X^p d\mu < C|\phi_i|_X^p$$

for each i, where C is a constant independent of n. Thus, $\langle u_n, \phi_1 \rangle$ is a uniformly bounded sequence in the reflexive space $L^p(S, \mathbb{R})$. Therefore, there exists a subsequence (n_1) and $\xi_1 \in L^p(S, \mathbb{R})$ such that

$$\int_S \langle u_{n_1}, \phi_1 \rangle \psi \, d\mu \to \int_S \xi_1 \psi \, d\mu \qquad \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R}).$$

Repeating the above process with each ϕ_i and subsequence obtained from previous step, there exists a subsequence (n_k) and $(\xi_i)_{i \in \mathbb{N}}$ such that

$$\int_S \langle u_{n_k}, \phi_i \rangle \psi \, d\mu \to \int_S \xi_i \psi \, d\mu \qquad \forall i \in \mathbb{N}, \ \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R}).$$

Finally, we can define $u \in L^p(S, X^*)$ by

$$\int_S \langle u, \phi_i \psi \rangle \, d\mu := \int_S \xi_i \psi \, d\mu \qquad \forall i \in \mathbb{N}, \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R})$$

and note that,

$$\int_S \langle u_{n_k}, \phi_i \rangle \psi \, d\mu \to \int_S \xi_i \psi \, d\mu = \int_S \langle u, \phi_i \rangle \psi \, d\mu \qquad \forall i \in \mathbb{N}, \ \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R})$$

as desired. $\square$

Chapter 2

Wiener driven SPDEs with locally monotone coefficients

In this chapter, we consider stochastic partial differential equations driven by Wiener noise when the coefficients satisfy local monotonicity condition. First, we obtain higher moment a priori estimates for solutions to such SPDEs under appropriate coercivity condition. The estimates so obtained are then used to extend the existence and uniqueness results of Liu and Röckner [30] for nonlinear SPDEs governed by locally monotone operators to allow derivatives in the operator acting on the solution under the stochastic integral. The coercivity condition which we have identified, is the natural condition given that higher order moment estimates are essential in proving existence of solution under the local monotonicity and growth conditions in [30]. The fact that our coercivity condition is the optimal one, is explained in Example 2.5 with the help of a result by Brzeźniak and Veraar [4]. Under this modified coercivity conditon, the results presented in this chapter can be applied to the SPDEs which do not fit the framework of [28, 30]. Examples of such SPDEs are given in Section 2.5. The work presented in this chapter is based on my joint article [36].

This chapter is organized as follows. In Section 2.1 the main results about higher-order moment estimates as well as existence and uniqueness of solutions are stated, together with the assumptions required. Section 2.2 is devoted to proving the a priori estimates and uniqueness of the solution. Galerkin discretization is used to obtain a finite-dimensional approximation to (2.1) in Section 2.3. Moreover moment bounds for the solutions of the finite-dimensional equations, uniform in the discretization parameter, are established. These are used in Section 2.4 to prove existence of solution to (2.1). Finally, Section 2.5 is devoted to examples of quasi-linear and semi-linear stochastic partial differential equations which fit into the framework of this chapter.

2.1 Assumptions and main results

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis and $W := (W_t)_{t\in[0,T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. Further, let

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

be a Gelfand triple.

Consider the stochastic evolution equation (SEE)

$$u_t = u_0 + \int_0^t A_s(u_s)ds + \sum_{j=1}^{\infty} \int_0^t B_s^j(u_s)dW_s^j, \quad t \in [0,T]. \tag{2.1}$$

Here, A and B^j, $j \in \mathbb{N}$ are assumed to be non-linear operators mapping $[0,T] \times \Omega \times V$ into V^* and H respectively. Further, we assume that for all $v, w \in V$, the processes $(\langle A_t(v), w\rangle)_{t\in[0,T]}$ and $((B_t^j(v), w))_{t\in[0,T]}$ are progressively measurable. Since by Pettis' theorem, the concept of weak measurability and strong measurability of a mapping coincide if the codomain is separable, we get that for all $v \in V$, $j \in \mathbb{N}$, $(A_t(v))_{t\in[0,T]}$ and $(B_t^j(v))_{t\in[0,T]}$ are progressively measurable. Finally, u_0 is assumed to be a given H-valued $\mathscr{F}_0$-measurable random variable.

The following assumptions are made on the operators. There exist constants $\alpha > 1$, $\beta \geq 0$, $p_0 \geq \beta + 2$, $\theta > 0$, K, L and a nonnegative $f \in L^{\frac{p_0}{2}}((0,T)\times\Omega;\mathbb{R})$ such that, almost surely, the following conditions hold for all $t \in [0,T]$.

A - 2.1 (Hemicontinuity)**.** For all $y, x, \bar{x}$ in V, the map

$$\varepsilon \mapsto \langle A_t(x+\varepsilon\bar{x}), y\rangle$$

is continuous.

A - 2.2 (Local Monotonicity[1])**.** For all $x, \bar{x}$ in V,

$$\begin{aligned}2\langle A_t(x) - A_t(\bar{x}), x-\bar{x}\rangle &+ \sum_{j=1}^{\infty} |B_t^j(x) - B_t^j(\bar{x})|_H^2 \\ &\leq L(1+|\bar{x}|_V^\alpha)(1+|\bar{x}|_H^\beta)|x-\bar{x}|_H^2.\end{aligned}$$

A - 2.3 (p_0-Stochastic Coercivity)**.** For all x in V,

$$2\langle A_t(x), x\rangle + (p_0-1)\sum_{j=1}^{\infty}|B_t^j(x)|_H^2 + \theta|x|_V^\alpha \leq f_t + K|x|_H^2.$$

A - 2.4 (Growth of A)**.** For all x in V,

$$|A_t(x)|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (f_t + K|x|_V^\alpha)(1+|x|_H^\beta).$$

Note that, if $p_0 = 2$, i.e. $\beta = 0$ and $L = 0$, then the conditions A-2.1 to A-2.4 reduce to corresponding ones used in Krylov and Rozovskii [28].

Remark 2.1. From Assumptions A-2.3 and A-2.4, we obtain

$$\sum_{j=1}^{\infty}|B_t^j(x)|_H^2 \leq C\big(1 + f_t^{\frac{p_0}{2}} + |x|_H^{p_0} + |x|_V^\alpha + |x|_V^\alpha|x|_H^\beta\big)$$

almost surely for all $t \in [0,T]$ and $x \in V$. Indeed, using Hölder's inequality, Young's inequality and Assumption A-2.4, we obtain that almost surely for all $x \in V$ and $t \in [0,T]$,

$$\begin{aligned}|\langle A_t(x), x\rangle| &\leq \frac{\alpha-1}{\alpha}|A_t(x)|_{V^*}^{\frac{\alpha}{\alpha-1}} + \frac{1}{\alpha}|x|_V^\alpha \\ &\leq \frac{\alpha-1}{\alpha}\big((f_t+K|x|_V^\alpha)(1+|x|_H^\beta)\big) + \frac{1}{\alpha}|x|_V^\alpha \\ &\leq C\Big(f_t + |x|_V^\alpha + |x|_V^\alpha|x|_H^\beta + f_t^{\frac{p_0}{2}} + (1+|x|_H)^{p_0}\Big).\end{aligned}$$

The above inequality along with Assumption A-2.3 gives,

$$\begin{aligned}(p_0-1)\sum_{j=1}^{\infty}|B_t^j(x)|_H^2 \leq& f_t + K|x|_H^2 - \theta|x|_V^\alpha + C\Big(f_t + |x|_V^\alpha + |x|_V^\alpha|x|_H^\beta + f_t^{\frac{p_0}{2}} + (1+|x|_H)^{p_0}\Big) \\ \leq& C\Big((1+f_t)^{\frac{p_0}{2}} + |x|_V^\alpha + |x|_V^\alpha|x|_H^\beta + f_t^{\frac{p_0}{2}} + (1+|x|_H)^{p_0}\Big)\end{aligned}$$

and hence the result.

[1] One may like to replace the constant L in Assumption A-2.2 by a function, say g_t. However we need that the solution u to (2.1) given by Definition 2.1 lies in the space Ψ (see Definition 2.2). For this to hold we would need g to be at least essentially bounded.

Further, in the case $p_0 = 2$, i.e. $\beta = 0$, using the similar argument as above, we get

$$\sum_{j=1}^{\infty} |B_t^j(x)|_H^2 \le C\Big(f_t + |x|_H^2 + |x|_V^\alpha\Big)$$

almost surely for all $t \in [0,T]$ and $x \in V$.

Remark 2.2. From Assumptions A-2.1, A-2.2 and A-2.4 we get that almost surely for all $t \in [0,T]$, the operator A_t is demicontinuous, i.e. $v_n \to v$ in V implies that $A_t(v_n) \rightharpoonup A_t(v)$ in V^*. This follows using similar arguments as in the proof of Lemma 2.1 in Krylov and Rozovskii [28]. Indeed, $(v_n)_{n\in\mathbb{N}}$ being convergent sequence is bounded. Thus for any subsequence $(v_{n_k})_{k\in\mathbb{N}}$, it follows from Assumption A-2.4 that almost surely for all $t \in [0,T]$, the sequence $(A_t(v_{n_k}))_{k\in\mathbb{N}}$ is bounded in V^*. Since V^* is a reflexive Banach space, there exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ and $a_t^\infty : \Omega \to V^*$ such that $A_t(v_{n_{k_l}}) \rightharpoonup a_t^\infty$ in V^* almost surely for all $t \in [0,T]$. It remains to show $a_t^\infty = A_t(v)$ almost surely for all $t \in [0,T]$. From Assumption A-2.2, it follows almost surely for all $t \in [0,T]$

$$\langle A_t(u) - A_t(v_{n_{k_l}}), u - v_{n_{k_l}}\rangle - L(1 + |v_{n_{k_l}}|_V^\alpha)(1 + |v_{n_{k_l}}|_H^\beta)|u - v_{n_{k_l}}|_H^2 \le 0,$$

for any $u \in V$. Taking limit $l \to \infty$, one obtains[2] almost surely for all $t \in [0,T]$

$$\langle A_t(u) - a_t^\infty, u - v\rangle - L(1 + |v|_V^\alpha)(1 + |v|_H^\beta)|u - v|_H^2 \le 0,$$

which on substituting $u = v + \lambda w$ for some $\lambda > 0$, $w \in V$ and then dividing by λ gives,

$$\langle A_t(v + \lambda w) - a_t^\infty, w\rangle - \lambda L(1 + |v|_V^\alpha)(1 + |v|_H^\beta)|w|_H^2 \le 0.$$

Finally, taking the limit $\lambda \to 0$ and using Assumption A-2.1, we obtain

$$\langle A_t(v) - a_t^\infty, w\rangle \le 0$$

almost surely for all $t \in [0,T]$. Changing w to $-w$, we get $\langle A_t(v) - a_t^\infty, w\rangle \ge 0$ and since $w \in V$ is arbitrary, we have $a_t^\infty = A_t(v)$ almost surely for all $t \in [0,T]$. Since the above result hold for any subsequence (v_{n_k}) of (v_n), we get[3] $A_t(v_n) \rightharpoonup A_t(v)$.

One consequence of Remark 2.2 is that, progressive measurability of some process $(v_t)_{t\in[0,T]}$ implies the progressive measurability of the process $(A_t(v_t))_{t\in[0,T]}$.

Definition 2.1 (Solution)**.** An adapted, continuous, H-valued process u is called a solution of the stochastic evolution equation (2.1) if

i) $dt \times \mathbb{P}$ almost everywhere $u \in V$ and

$$\mathbb{E}\int_0^T (|u_t|_V^\alpha + |u_t|_H^2)\, dt < \infty\,,$$

ii) for every $t \in [0,T]$ and $\phi \in V$, almost surely[4]

$$(u_t, \phi) = (u_0, \phi) + \int_0^t \langle A_s(u_s), \phi\rangle ds + \sum_{j=1}^{\infty}\int_0^t (\phi, B_s^j(u_s))dW_s^j\,.$$

Note that the fact that u is a continuous, H-valued process and i) in Definition 2.1 implies

[2] Note that $v_n \to v$ in V and $f_n \rightharpoonup f$ in V^* implies $\langle f_n, v_n\rangle \to \langle f, v\rangle$.

[3] Note that if every subsequence of (x_n) has a further subsequence (x_{n_k}) which converges to x, then whole sequence (x_n) converges to x.

[4] Note that from Assumptions A-2.2 and A-2.4, for each $i \in \mathbb{N}$, $B_t^i(\cdot) : V \to H$ is strongly continuous, almost surely for all $t \in [0,T]$. Thus, progressive measurability of some process $(v_t)_{t\in[0,T]}$ implies the progressive measurability of the process $(B_t(v_t))_{t\in[0,T]}$. Further, in view of A-2.3, A-2.4 along with (i), integrals in (ii) are well defined.

that almost surely,

$$\int_0^T \Big(|u_t|_H^\beta + |u_t|_V^\alpha |u_t|_H^\beta\Big)\, dt < \infty.$$

The following are the main results of this chapter.

Theorem 2.1 (A priori estimates)**.** *If u is a solution of* (2.1) *and Assumptions A-2.3 and A-2.4 hold, then*

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}|u_t|_H^{p_0} + \mathbb{E}\int_0^T |u_t|_H^{p_0-2}|u_t|_V^\alpha dt \leq C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\, ds\Big) \quad \textit{for } p_0 > 2,\\ \sup_{t\in[0,T]} \mathbb{E}|u_t|_H^2 + \mathbb{E}\int_0^T |u_t|_V^\alpha dt \leq C\mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s ds\Big).\end{aligned} \tag{2.2}$$

Moreover,

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]} |u_t|_H^2 \leq C\mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s ds\Big)\\ \textit{and} \qquad &\mathbb{E}\sup_{t\in[0,T]} |u_t|_H^{p_0 r} \leq C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\, ds\Big)^r,\end{aligned} \tag{2.3}$$

for any $r \in (0,1)$, where C depends only on p_0, K, T, r and θ.

Note that if $p_0 > 2$ then one cannot make use of the Burkholder–Davis–Gundy inequality to prove (2.3). Indeed, in this case one would end up with a higher moment on the right-hand side than on the left when trying to prove the a priori estimate. One avoids this problem by using Lenglart's inequality (see, e.g. Lemma 3.2 in Gyöngy and Krylov [14]) but this means one can only get (2.3) for $2 \leq p < p_0$.

Theorem 2.2 (Uniqueness of solution)**.** *Let Assumptions A-2.2, A-2.3 and A-2.4 hold and $u_0, \bar{u}_0 \in L^{p_0}(\Omega; H)$. If u and $\bar{u}$ are two solutions of* (2.1) *with $u_0 = \bar{u}_0$ $\mathbb{P}$-a.s., then the processes u and $\bar{u}$ are indistinguishable, i.e.*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} |u_t - \bar{u}_t|_H = 0\Big) = 1.$$

Theorem 2.3 (Existence of solution)**.** *If Assumptions A-2.1 to A-2.4 hold and $u_0 \in L^{p_0}(\Omega; H)$, then the SEE* (2.1) *has a unique solution.*

At first glance Assumption A-2.3 (equivalently $\tilde{\text{A}}$-3, see the Appendix) seems to be more restrictive than the one used in Liu and Röckner [30] and the reader may conclude that our results do not cover some SPDEs that can be treated by [30]. However this is not the case. Given the growth condition on operator B that has been assumed in [30, Theorem 1.1, inequality (1.2)], Assumption $\tilde{\text{A}}$-3 follows immediately from their coercivity condition. Indeed, below we recall the coercivity condition (H3) and growth condition (1.2) used by Liu and Röckner [30]: for all $(t,\omega) \in [0,T] \times \Omega$ and $x \in V$,

$$2\langle A_t(x), x\rangle + |B_t(x)|_{L_2(U,H)}^2 + \theta|x|_V^\alpha \leq f_t + K|x|_H^2 \tag{2.4}$$

and

$$|B_t(x)|_{L_2(U,H)}^2 \leq C(f_t + |x|_H^2). \tag{2.5}$$

Then multiplying (2.5) by $(p_0 - 2)$ and adding the equation obtained to (2.4), we get

$$2\langle A_t(x), x\rangle + (p_0-1)|B_t(x)|_{L_2(U,H)}^2 + \theta|x|_V^\alpha \leq \tilde{f}_t + \tilde{K}|x|_H^2$$

where, $\tilde{f}_t = f_t + C(p_0-2)f_t$ with $\tilde{f} \in L^{\frac{p_0}{2}}((0,T)\times\Omega;\mathbb{R})$ and $\tilde{K} = K + C(p_0-2)$ which implies $\tilde{\text{A}}$-3 holds. Examples 2.1, 2.2, 2.3 and 2.4 show that the converse does not hold. Moreover Example 2.5 shows that our Assumption A-2.3 (which is equivalent to $\tilde{\text{A}}$-3) is sharp.

2.2 A priori estimates and uniqueness of solution

Proof of Theorem 2.1. Let u be a solution to equation (2.1) in the sense of Definition 2.1. Then, applying the Itô's formula for the square of the norm (see, e.g., [28, Chapter 1, Theorem 3.2] or [40, Theorem 4.2.5]), we obtain

$$|u_t|_H^2 = |u_0|_H^2 + \int_0^t \Big(2\langle A_s(u_s), u_s\rangle + \sum_{j=1}^\infty |B_s^j(u_s)|_H^2\Big) ds + 2\sum_{j=1}^\infty \int_0^t (u_s, B_s^j(u_s)) dW_s^j \tag{2.6}$$

almost surely for all $t \in [0,T]$. Notice that this is a real-valued Itô process. Thus, by Itô's formula,

$$\begin{aligned} d|u_t|_H^{p_0} =& \frac{p_0}{2}|u_t|_H^{p_0-2}\Big(2\langle A_t(u_t), u_t\rangle + \sum_{j=1}^\infty |B_t^j(u_t)|_H^2\Big)\, dt \\ &+ p_0|u_t|_H^{p_0-2}\sum_{j=1}^\infty (u_t, B_t^j(u_t))dW_t^j + \frac{p_0(p_0-2)}{2}|u_t|_H^{p_0-4}\sum_{j=1}^\infty |(u_t, B_t^j(u_t))|^2\, dt \end{aligned}$$

almost surely for all $t \in [0,T]$, which on using Cauchy-Schwarz inequality gives

$$\begin{aligned} d|u_t|_H^{p_0} \le \frac{p_0}{2}|u_t|_H^{p_0-2}\Big(2\langle A_t(u_t), u_t\rangle + (p_0-1)\sum_{j=1}^\infty |B_t^j(u_t)|_H^2\Big)dt& \\ + p_0|u_t|_H^{p_0-2}\sum_{j=1}^\infty (u_t, B_t^j(u_t))dW_t^j\, .& \end{aligned} \tag{2.7}$$

We aim to apply Lemma 1.9. To that end let τ be some stopping time. Moreover, to estimate the term containing the stochastic integral in (2.7), we define for each $n \in \mathbb{N}$

$$\sigma_n := \inf\{t \in [0,T] : |u_t|_H > n\} \wedge T. \tag{2.8}$$

Then, $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of stopping times converging to T as $n \to \infty$. By using Assumption A-2.3 and Young's inequality in (2.7), we obtain

$$\begin{aligned} &|u_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} \\ &\quad \le |u_0|_H^{p_0} + \frac{p_0}{2}\int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0-2}\Big(f_s + K|u_s|_H^2 - \theta|u_s|_V^\alpha\Big)ds \\ &\qquad + p_0\sum_{j=1}^\infty \int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0-2}(u_s, B_s^j(u_s))dW_s^j \\ &\quad \le |u_0|_H^{p_0} + \int_0^{t\wedge\sigma_n\wedge\tau} f_s^{\frac{p_0}{2}}\, ds + \frac{p_0-2}{2}\int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0} ds + \frac{p_0}{2}K\int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0} ds \\ &\qquad - \theta\frac{p_0}{2}\int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0-2}|u_s|_V^\alpha ds + p_0\sum_{j=1}^\infty \int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0-2}(u_s, B_s^j(u_s))dW_s^j. \end{aligned}$$

Thus,

$$\begin{aligned} &|u_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} + \theta\frac{p_0}{2}\int_0^{t\wedge\sigma_n\wedge\tau} |u_s|_H^{p_0-2}|u_s|_V^\alpha ds \\ &\qquad \le |u_0|_H^{p_0} + \int_0^{t\wedge\sigma_n\wedge\tau} f_s^{\frac{p_0}{2}}\, ds + \frac{p_0(K+1)-2}{2}\int_0^t \mathbf{1}_{\{s\le\sigma_n\wedge\tau\}}|u_s|_H^{p_0} ds \\ &\qquad + p_0\sum_{j=1}^\infty \int_0^{t\wedge\sigma_n} \mathbf{1}_{\{s\le\tau\}}|u_s|_H^{p_0-2}(u_s, B_s^j(u_s))dW_s^j. \end{aligned} \tag{2.9}$$

Then in view of Remark 2.1 and the fact that u is a solution of equation (2.1), it follows that

$$\mathbb{E}\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}\mathbf{1}_{\{s\leq\tau\}}|u_s|_H^{p_0-2}(u_s,B_s^j(u_s))dW_s^j=0.$$

Therefore, taking expectation in (2.9), we obtain

$$\begin{aligned}\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau}|_H^{p_0}&+\theta\frac{p_0}{2}\mathbb{E}\int_0^{t\wedge\sigma_n\wedge\tau}|u_s|_H^{p_0-2}|u_s|_V^{\alpha}ds\\&\leq\mathbb{E}|u_0|_H^{p_0}+\mathbb{E}\int_0^T f_s^{\frac{p_0}{2}}ds+\frac{p_0(K+1)-2}{2}\mathbb{E}\int_0^t|u_{s\wedge\sigma_n\wedge\tau}|_H^{p_0}ds.\end{aligned}\tag{2.10}$$

From this Gronwall's lemma yields

$$\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau}|_H^{p_0}\leq e^{\frac{p_0(K+1)-2}{2}T}\mathbb{E}\Big(|u_0|_H^{p_0}+\int_0^T f_s^{\frac{p_0}{2}}ds\Big)\tag{2.11}$$

for all $t\in[0,T]$. Letting $n\to\infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}|u_{t\wedge\tau}|_H^{p_0}\leq e^{\frac{p_0(K+1)-2}{2}T}\mathbb{E}\Big(|u_0|_H^{p_0}+\int_0^T f_s^{\frac{p_0}{2}}ds\Big)$$

for all $t\in[0,T]$. Using Lemma 1.9, we get

$$\mathbb{E}\sup_{t\in[0,T]}|u_t|_H^{p_0r}\leq\frac{2-r}{1-r}e^{\frac{p_0(K+1)-2}{2}rT}\mathbb{E}\Big(|u_0|_H^{p_0}+\int_0^T f_s^{\frac{p_0}{2}}ds\Big)^r$$

for any $r\in(0,1)$, which proves second inequality in (2.3).

In order to prove (2.2), we use the estimate (2.11) in the right-hand side of (2.10) with $\tau=T$ and with $n\to\infty$. We thus obtain,

$$\mathbb{E}|u_t|_H^{p_0}+\theta\frac{p_0}{2}\mathbb{E}\int_0^t|u_s|_H^{p_0-2}|u_s|_V^{\alpha}ds\leq C\mathbb{E}\Big(|u_0|_H^{p_0}+\int_0^T f_s^{\frac{p_0}{2}}ds\Big)$$

for all $t\in[0,T]$. If Assumption A-2.3 holds for some $p_0\geq\beta+2$, then it holds for $p_0=2$ as well. Thus, using the stopping times $(\sigma_n)_{n\in\mathbb{N}}$ in (2.6) and taking expectation, we obtain, using the same localizing argument as before, that

$$\mathbb{E}|u_t|_H^2+\theta\mathbb{E}\int_0^t|u_s|_V^{\alpha}ds\leq\mathbb{E}\Big(|u_0|_H^2+\int_0^T f_sds\Big)+\mathbb{E}\int_0^t K|u_s|_H^2ds,$$

for all $t\in[0,T]$. Application of Gronwall's lemma yields,

$$\sup_{t\in[0,T]}\mathbb{E}|u_t|_H^2\leq C\mathbb{E}\Big(|u_0|_H^2+\int_0^T f_sds\Big)$$

which in turn gives

$$\theta\mathbb{E}\int_0^T|u_s|_V^{\alpha}ds\leq C\mathbb{E}\Big(|u_0|_H^2+\int_0^T f_s\,ds\Big)$$

and hence (2.2) holds.

To complete the proof it remains to show first inequality in (2.3). This is done using the same argument as in Krylov and Rozovskii [28]. It is included here for convenience of the reader. Considering the sequence of stopping times (σ_n) defined in (2.8) and using Remark 2.1 along with Definition 2.1, we observe that the stochastic integral in the right-hand side of (2.6) is a local martingale. Thus invoking the Burkholder–Davis–Gundy inequality and then using

Cauchy–Schwarz inequality, Remark 2.1 and Young's inequality we obtain,

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}(u_s,B_s^j(u_s))dW_s^j\Big| &\le 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|(u_s,B_s^j(u_s))|^2ds\Big)^{\frac{1}{2}}\\
&\le 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|u_s|_H^2|B_s^j(u_s)|_H^2ds\Big)^{\frac{1}{2}}\\
&\le 4\mathbb{E}\Big(\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\int_0^{T\wedge\sigma_n}\big(f_s+|u_s|_H^2+|u_s|_V^{\alpha}\big)ds\Big)^{\frac{1}{2}}\\
&\le \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2+C\mathbb{E}\int_0^{T\wedge\sigma_n}\big(f_s+|u_s|_H^2+|u_s|_V^{\alpha}\big)ds.
\end{aligned}
\tag{2.12}
$$

We now replace t by $t\wedge\sigma_n$ in (2.6) and take supremum and then expectation. Then using Assumption A-2.3 along with (2.12), we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\\
\le&\epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2+C\Big(\mathbb{E}|u_0|_H^2+\mathbb{E}\int_0^T f_s\,ds+\mathbb{E}\int_0^T|u_s|_V^{\alpha}ds+\sup_{t\in[0,T]}\mathbb{E}|u_t|_H^2\Big).
\end{aligned}
$$

Finally, by choosing ϵ small and using (2.2) for $p_0=2$, we obtain

$$
\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\le C\Big(\mathbb{E}|u_0|_H^2+\mathbb{E}\int_0^T f_s\,ds\Big)
$$

which on allowing $n\to\infty$ and using Fatou's lemma finishes the proof. □

Definition 2.2. Let Ψ be defined as the collection of V-valued and $\mathscr{F}_t$-adapted processes ψ satisfying

$$
\int_0^T\rho(\psi_s)ds<\infty\quad\text{a.s.}\,,
$$

where

$$
\rho(x):=L(1+|x|_V^{\alpha})(1+|x|_H^{\beta})
$$

for all $x\in V$.

Note that if u is a solution to (2.1), then $u\in\Psi$.

Remark 2.3. For any $\psi\in\Psi$ and $v\in L^2(\Omega,C([0,T];H))$,

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\rho(\psi_s)|v_s|_H^2ds\Big] &\le \mathbb{E}\sup_{s\in[0,t]}|v_s|_H^2\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\rho(\psi_s)ds\\
&=\mathbb{E}\sup_{s\in[0,t]}|v_s|_H^2[1-e^{-\int_0^t\rho(\psi_r)dr}]\\
&\le \mathbb{E}\sup_{s\in[0,t]}|v_s|_H^2<\infty.
\end{aligned}
$$

This remark justifies the existence of the bounded variation integrals appearing in the proof of uniqueness that follows.

Proof of Theorem 2.2. Consider two solutions u and $\bar{u}$ of (2.1). Thus,

$$
u_t-\bar{u}_t=u_0-\bar{u}_0+\int_0^t\left(A_s(u_s)-A_s(\bar{u}_s)\right)\,ds+\sum_{j=1}^{\infty}\int_0^t\left(B_s^j(u_s)-B_s^j(\bar{u}_s)\right)\,dW_s^j \tag{2.13}
$$

almost surely for all $t \in [0,T]$. Using the Itô's formula and the product rule, we obtain

$$
\begin{aligned}
d\Big(e^{-\int_0^t \rho(\bar u_s)\,ds}|u_t-\bar u_t|_H^2\Big) &= e^{-\int_0^t \rho(\bar u_s)ds}\big[d|u_t-\bar u_t|_H^2-\rho(\bar u_t)|u_t-\bar u_t|_H^2\,dt\big]\\
= e^{-\int_0^t \rho(\bar u_s)ds}\bigg[&\Big(2\langle A_t(u_t)-A_t(\bar u_t),u_t-\bar u_t\rangle+\sum_{j=1}^\infty|B_t^j(u_t)-B_t^j(\bar u_t)|_H^2\Big)\,dt\\
&+\sum_{j=1}^\infty 2\big(u_t-\bar u_t,B_t^j(u_t)-B_t^j(\bar u_t)\big)dW_t^j-\rho(\bar u_t)|u_t-\bar u_t|_H^2dt\bigg]
\end{aligned}
\tag{2.14}
$$

almost surely for all $t \in [0,T]$. For each $n \in \mathbb{N}$, we define the sequence (σ_n) of stopping times as follows.

$$
\sigma_n := \inf\{t\in[0,T] : |u_t|_H > n\}\wedge\inf\{t\in[0,T] : |\bar u_t|_H > n\}\wedge T. \tag{2.15}
$$

Using Assumption A-2.2 and replacing t by $t_n := t\wedge\sigma_n$ in (2.14), we get

$$
e^{-\int_0^{t_n}\rho(\bar u_s)ds}|u_{t_n}-\bar u_{t_n}|_H^2-|u_0-\bar u_0|_H^2 \le 2\sum_{j=1}^\infty\int_0^{t_n} e^{-\int_0^s\rho(\bar u_r)dr}\big(u_s-\bar u_s,B_s^j(u_s)-B_s^j(\bar u_s)\big)dW_s^j.
$$

Thus if $u_0=\bar u_0$ $\mathbb{P}$-a.s., we have

$$
\mathbb{E}[e^{-\int_0^{t_n}\rho(\bar u_s)ds}|u_{t_n}-\bar u_{t_n}|_H^2]\le 0.
$$

Letting $n\to\infty$ and using Fatou's lemma, we obtain that $\mathbb{P}(|u_t-\bar u_t|_H^2=0)=1$ for all $t\in[0,T]$. This, together with the continuity of $u-\bar u$ in H, concludes the proof. □

We can have some results about the continuous dependence of the solution to (2.1) on the initial data if we assume the following.

A - 2.5 (Strong Monotonicity)**.** There exist constants $\theta'>0$, K such that almost surely, for all $t\in[0,T]$ and $x,\bar x\in V$,

$$
2\langle A_t(x)-A_t(\bar x),x-\bar x\rangle+(p_0-1)\sum_{j=1}^\infty|B_t^j(x)-B_t^j(\bar x)|_H^2\le-\theta'|x-\bar x|_V^\alpha+K|x-\bar x|_H^2\,.
$$

A - 2.6. There exist a constant K such that almost surely, for all $t\in[0,T]$ and $x,\bar x$ in V,

$$
\sum_{j=1}^\infty|B_t^j(x)-B_t^j(\bar x)|_H^2\le K\Big(|x-\bar x|_H^2+|x-\bar x|_V^\alpha\Big).
$$

If we replace the local monotonicity Assumption A-2.2 in Theorem 2.2 by the strong monotonicity Assumption A-2.5, then we obtain the following result about the continuous dependence of the solution to (2.1) on the initial data.

Theorem 2.4. *Let Assumptions A-2.3 to A-2.5 hold and $u_0,\bar u_0\in L^{p_0}(\Omega;H)$. If u and $\bar u$ are two solutions of* (2.1) *with initial condition u_0 and $\bar u_0$ respectively, then for $p_0>2$*

$$
\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}|u_t-\bar u_t|_H^{p_0}+\mathbb{E}\int_0^T|u_t-\bar u_t|_H^{p_0-2}|u_t-\bar u_t|_V^\alpha dt &< C\mathbb{E}|u_0-\bar u_0|_H^{p_0}\,,\\
\mathbb{E}\sup_{t\in[0,T]}|u_t-\bar u_t|_H^{p_0r} &< C\mathbb{E}|u_0-\bar u_0|_H^{p_0r}\,,
\end{aligned}
$$

for any $r\in(0,1)$ and

$$
\sup_{t\in[0,T]}\mathbb{E}|u_t-\bar u_t|_H^2+\mathbb{E}\int_0^T|u_t-\bar u_t|_V^\alpha dt<C\mathbb{E}|u_0-\bar u_0|_H^2\,.
$$

Proof. The proof is very similar to the proof of Theorem 2.1. Indeed we apply Itô's formula from [28] to (2.13) and repeat the proof of Theorem 2.1 for the process $u_t-\bar u_t$. Here we note that one needs to use the strong monotonicity Assumption A-2.5 in place of Assumption A-2.3 and work with the sequence of stopping times given by (2.15). However, we include the proof for the convenience of reader.

Let u and $\bar{u}$ be two solutions of (2.1) in the sense of Definition 2.1 so that (2.13) holds almost surely for all $t \in [0,T]$. Applying Itô's formula for the square of the norm to (2.13), we get almost surely for all $t \in [0,T]$,

$$
\begin{aligned}
|u_t - \bar{u}_t|_H^2 = |u_0 - \bar{u}_0|_H^2 + \int_0^t \Big(& 2\langle A_s(u_s) - A_s(\bar{u}_s), u_s - \bar{u}_s \rangle \\
+ \sum_{j=1}^{\infty} |B_s^j(u_s) - B_s^j(\bar{u}_s)|_H^2 \Big) ds & + 2 \sum_{j=1}^{\infty} \int_0^t (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j ,
\end{aligned}
\tag{2.16}
$$

which is a real-valued Itô process. Thus, by Itô's formula and using Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
|u_t - \bar{u}_t|_H^{p_0} \leq |u_0 - \bar{u}_0|_H^{p_0} + \frac{p_0}{2} \int_0^t & |u_s - \bar{u}_s|_H^{p_0-2} \Big(2\langle A_s(u_s) - A_s(\bar{u}_s), u_s - \bar{u}_s \rangle \\
& + \sum_{j=1}^{\infty} (p_0 - 1) |B_s^j(u_s) - B_s^j(\bar{u}_s)|_H^2 \Big) ds \\
& + p_0 \int_0^t |u_s - \bar{u}_s|_H^{p_0-2} \sum_{j=1}^{\infty} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j
\end{aligned}
\tag{2.17}
$$

almost surely for all $t \in [0,T]$. In order to apply Lemma 1.9, we consider a bounded stopping time τ and the sequence of stopping times (σ_n) defined in (2.15). Replacing t by $t \wedge \sigma_n \wedge \tau$ in (2.17) and using Assumption A-2.5, we get

$$
\begin{aligned}
& |u_{t\wedge\sigma_n\wedge\tau} - \bar{u}_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} \\
\leq & |u_0 - \bar{u}_0|_H^{p_0} + \frac{p_0}{2} \int_0^{t\wedge\sigma_n\wedge\tau} |u_s - \bar{u}_s|_H^{p_0-2} \Big(K |u_s - \bar{u}_s|_H^2 - \theta' |u_s - \bar{u}_s|_V^{\alpha} \Big) ds \\
& + p_0 \sum_{j=1}^{\infty} \int_0^{t\wedge\sigma_n\wedge\tau} |u_s - \bar{u}_s|_H^{p_0-2} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j \\
\leq & |u_0 - \bar{u}_0|_H^{p_0} + \int_0^{t\wedge\sigma_n\wedge\tau} \Big(K \frac{p_0}{2} |u_s - \bar{u}_s|_H^{p_0} - \theta' \frac{p_0}{2} |u_s - \bar{u}_s|_H^{p_0-2} |u_s - \bar{u}_s|_V^{\alpha} \Big) ds \\
& + p_0 \sum_{j=1}^{\infty} \int_0^{t\wedge\sigma_n} \mathbf{1}_{\{s \leq \tau\}} |u_s - \bar{u}_s|_H^{p_0-2} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j
\end{aligned}
\tag{2.18}
$$

almost surely for all $t \in [0,T]$ and $n \in \mathbb{N}$. Then in view of Remark 2.1 and the fact that u and $\bar{u}$ are solutions of equation (2.1), it follows that for all $t \in [0,T]$ and $n \in \mathbb{N}$,

$$
\mathbb{E} \sum_{j=1}^{\infty} \int_0^{t\wedge\sigma_n} \mathbf{1}_{\{s \leq \tau\}} |u_s - \bar{u}_s|_H^{p_0-2} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j = 0 .
$$

Therefore, taking expectation in (2.18), we obtain for all $t \in [0,T]$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau} - \bar{u}_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} + \theta' \frac{p_0}{2} \mathbb{E} \int_0^{t\wedge\sigma_n\wedge\tau} & |u_s - \bar{u}_s|_H^{p_0-2} |u_s - \bar{u}_s|_V^{\alpha} ds \\
\leq \mathbb{E}|u_0 - \bar{u}_0|_H^{p_0} + K \frac{p_0}{2} \mathbb{E} \int_0^{t\wedge\sigma_n\wedge\tau} & |u_s - \bar{u}_s|_H^{p_0} ds \\
\leq \mathbb{E}|u_0 - \bar{u}_0|_H^{p_0} + C \mathbb{E} \int_0^t & |u_{s\wedge\sigma_n\wedge\tau} - \bar{u}_{s\wedge\sigma_n\wedge\tau}|_H^{p_0} ds
\end{aligned}
\tag{2.19}
$$

From this Gronwall's lemma yields,

$$\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau} - \bar{u}_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} \le C\mathbb{E}|u_0 - \bar{u}_0|_H^{p_0} \tag{2.20}$$

for all $t \in [0,T]$ and $n \in \mathbb{N}$. Letting $n \to \infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}|u_{t\wedge\tau} - \bar{u}_{t\wedge\tau}|_H^{p_0} \le C\mathbb{E}|u_0 - \bar{u}_0|_H^{p_0}$$

for all $t \in [0,T]$. Using Lemma 1.9, we get

$$\mathbb{E}\sup_{t\in[0,T]}|u_t - \bar{u}_t|_H^{p_0 r} \le \frac{2-r}{1-r}C\mathbb{E}|u_0 - \bar{u}_0|_H^{p_0 r}$$

for any $r \in (0,1)$. Further, using the estimate (2.20) in the right-hand side of (2.19) with $\tau = T$ and with $n \to \infty$, we obtain

$$\mathbb{E}|u_t - \bar{u}_t|_H^{p_0} + \theta'\frac{p_0}{2}\mathbb{E}\int_0^t |u_s - \bar{u}_s|_H^{p_0-2}|u_s - \bar{u}_s|_V^{\alpha} ds \le C\mathbb{E}|u_0 - \bar{u}_0|_H^{p_0}$$

for all $t \in [0,T]$ as desired. Further, we note that if Assumption A-2.5 holds for some $p_0 \ge \beta+2$, then it holds for $p_0 = 2$ as well. Thus, from (2.16) we obtain

$$\mathbb{E}|u_t - \bar{u}_t|_H^2 + \theta'\mathbb{E}\int_0^t |u_s - \bar{u}_s|_V^{\alpha} ds \le \mathbb{E}|u_0 - \bar{u}_0|_H^2 + K\mathbb{E}\int_0^t |u_s - \bar{u}_s|_H^2 ds$$

for all $t \in [0,T]$. Application of Gronwall's lemma yields,

$$\sup_{t\in[0,T]}\mathbb{E}|u_t - \bar{u}_t|_H^2 \le C\mathbb{E}|u_0 - \bar{u}_0|_H^2\,, \tag{2.21}$$

which in turn gives,

$$\theta'\mathbb{E}\int_0^T |u_s - \bar{u}_s|_V^{\alpha} ds \le C\mathbb{E}|u_0 - \bar{u}_0|_H^2 \tag{2.22}$$

and hence the result. □

Remark 2.4. Assuming Assumption A-2.6 in addtion to assumptions made in Theorem 2.4 above, we further obtain

$$\mathbb{E}\sup_{t\in[0,T]}|u_t - \bar{u}_t|_H^2 < C\mathbb{E}|u_0 - \bar{u}_0|_H^2\,.$$

Indeed, by considering the sequence of stopping times σ_n defined in (2.15) and using Remark 2.1 along with Definition 2.1, we observe that the stochastic integrals appearing in the right-hand side of (2.16) are martingales for each $n \in \mathbb{N}$. Thus using the Burkholder–Davis–Gundy inequality, Cauchy–Schwarz inequality along with Assumption A-2.6 and Young's inequality yields for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}(u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s))dW_s^j\Big| \\
&\qquad \le 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|(u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s))|^2 ds\Big)^{\frac{1}{2}} \\
&\qquad \le 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|u_s - \bar{u}_s|_H^2|B_s^j(u_s) - B_s^j(\bar{u}_s)|_H^2 ds\Big)^{\frac{1}{2}} \qquad (2.23)\\
&\qquad \le K\mathbb{E}\Big(\sup_{t\in[0,T]}|u_{t\wedge\sigma_n} - \bar{u}_{t\wedge\sigma_n}|_H^2\int_0^{T\wedge\sigma_n}\big(|u_s - \bar{u}_s|_H^2 + |u_s - \bar{u}_s|_V^{\alpha}\big)ds\Big)^{\frac{1}{2}} \\
&\qquad \le \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n} - \bar{u}_{t\wedge\sigma_n}|_H^2 + C\mathbb{E}\int_0^{T\wedge\sigma_n}\big(|u_s - \bar{u}_s|_H^2 + |u_s - \bar{u}_s|_V^{\alpha}\big)ds\,.
\end{aligned}$$

Moreover, replacing t by $t\wedge\sigma_n$, taking supremum and then expectation in (2.16) and then using Assumption A-2.5 and (2.23), we obtain for each $n \in \mathbb{N}$,

$$\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2 \leq \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2$$
$$+ C\Big(\mathbb{E}|u_0-\bar{u}_0|_H^2 + \mathbb{E}\int_0^T |u_s-\bar{u}_s|_V^\alpha ds + \sup_{t\in[0,T]}\mathbb{E}|u_t-\bar{u}_t|_H^2\Big).$$

Finally, by choosing ϵ small and using (2.21) and (2.22), we obtain for each $n \in \mathbb{N}$

$$\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2 \leq C\Big(\mathbb{E}|u_0-\bar{u}_0|_H^2\Big)$$

which on allowing $n\to\infty$ and using Fatou's lemma gives the desired result.

2.3 A priori estimates for Galerkin discretization

We show the existence of solution to SEE (2.1) using the Galerkin method. Consider a Galerkin scheme $(V_m)_{m\in\mathbb{N}}$ for V, i.e. for each $m\in\mathbb{N}$, V_m is an m-dimensional subspace of V such that $V_m \subset V_{m+1} \subset V$ and $\cup_{m\in\mathbb{N}}V_m$ is dense in V. Let $\{\phi_i : i = 1,2,\dots m\}$ be a basis of V_m. Assume that for each $m\in\mathbb{N}$, u_0^m is a V_m-valued $\mathscr{F}_0$-measurable random variable satisfying[5],

$$\sup_{m\in\mathbb{N}}\mathbb{E}|u_0^m|_H^{p_0} < \infty \;\text{ and }\; \mathbb{E}|u_0^m-u_0|_H^2 \to 0 \;\text{ as }\; m\to\infty. \tag{2.24}$$

For each $m\in\mathbb{N}$ and $\phi_i\in V_m$, $i=1,2,\dots,m$, consider the stochastic differential equation:

$$(u_t^m,\phi_i) = (u_0^m,\phi_i) + \int_0^t \langle A_s(u_s^m),\phi_i\rangle ds + \sum_{j=1}^m\int_0^t (\phi_i, B_s^j(u_s^m))dW_s^j \tag{2.25}$$

almost surely for all $t\in[0,T]$. Using the results on solvability of stochastic differential equations in finite dimensional space (see, e.g., [28, Chapter 2, Theorem 3.1]), together with Assumptions A-2.1 to A-2.4 and Remark 2.2, there exists a unique adapted and continuous (and thus progressively measurable) V_m-valued process u^m satisfying (2.25).

Lemma 2.1 (A priori Estimates for Galerkin Discretization)**.** *Suppose that* (2.24) *and Assumptions A-2.3 and A-2.4 hold. Then there exists a constant C independent of m, such that*

$$\sup_{t\in[0,T]}\mathbb{E}|u_t^m|_H^{p_0} + \mathbb{E}\int_0^T |u_t^m|_V^\alpha\,dt + \mathbb{E}\int_0^T |u_t^m|_H^{p_0-2}|u_t^m|_V^\alpha\,dt \leq C, \tag{2.26}$$

$$\mathbb{E}\sup_{t\in[0,T]}|u_t^m|_H^p \leq C, \tag{2.27}$$

with $p=2$ in case $p_0=2$ (i.e. $\beta=0$) and $p\in[2,p_0)$ if $p_0>2$. Further, we have

$$\mathbb{E}\int_0^T |A_s(u_s^m)|_{V^*}^{\frac{\alpha}{\alpha-1}}\,ds \leq C \tag{2.28}$$

$$\text{and}\quad \mathbb{E}\sum_{j=1}^\infty\int_0^T |B_s^j(u_s^m)|_H^2 ds \leq C. \tag{2.29}$$

Proof. Proof of (2.26) and (2.27) is almost a repetition of the proof of analogous estimates in Theorem 2.1. Indeed, for each $m\in\mathbb{N}$, we can define a sequence of stopping times

$$\sigma_n^m := \inf\{t\in[0,T] : |u_t^m|_H > n\}\wedge T$$

[5]We can always obtain such an approximating sequence. For example, consider $\{\phi_i\}_{i\in\mathbb{N}}\subset V$ forming an orthonormal basis in H and for each $m\in\mathbb{N}$, take $u_0^m = \Pi_m u_0$ where $\Pi_m : H\to V_m$ are the projection operators.

and repeat the steps of Theorem 2.1 by replacing u_t with u_t^m and σ_n with σ_n^m. There are two main points to be noted. The first is that the stochastic integral appearing on right-hand side of (2.6), with u_t replaced by u_t^m, is a local martingale for each $m \in \mathbb{N}$. Indeed, on a finite dimensional space, all norms are equivalent and hence

$$\mathbb{E}\int_0^{T\wedge\sigma_n^m} |u_t^m|_V^\alpha dt \le C_m \mathbb{E}\int_0^{T\wedge\sigma_n^m} n^\alpha dt < \infty$$

with some constant C_m. The second point is that, since

$$\sup_{m\in\mathbb{N}} \mathbb{E}|u_0^m|^{p_0} < \infty,$$

we can find a constant independent of m to obtain (2.26) and (2.27). The estimates (2.28) and (2.29) can be proved as below. Using Assumption A-2.4, we obtain

$$\begin{aligned} I &:= \mathbb{E}\int_0^T |A_s(u_s^m)|_{V^*}^{\frac{\alpha}{\alpha-1}} ds \le \mathbb{E}\int_0^T (f_s + K|u_s^m|_V^\alpha)(1+|u_s^m|_H^\beta) ds \\ &= \mathbb{E}\int_0^T f_s\, ds + \mathbb{E}\int_0^T f_s |u_s^m|_H^\beta ds + K\mathbb{E}\int_0^T |u_s^m|_V^\alpha ds + K\mathbb{E}\int_0^T |u_s^m|_V^\alpha |u_s^m|_H^\beta\, ds\,. \end{aligned}$$

Using Young's inequality, we see that

$$\begin{aligned} f_s + f_s|u_s^m|_H^\beta &\le \frac{4}{p_0} f_s^{\frac{p_0}{2}} + \frac{p_0-2}{p_0} + \frac{p_0-2}{p_0}|u_s^m|_H^{\beta\frac{p_0}{p_0-2}} \\ &\le \frac{4}{p_0} f_s^{\frac{p_0}{2}} + \frac{p_0-2}{p_0} + \frac{p_0-2-\beta}{p_0-2} + \frac{p_0-2}{p_0}|u_s^m|_H^{p_0} \end{aligned}$$

where we have used the fact $p_0 \ge \beta + 2$. This also implies $|u_s^m|_H^\beta \le (1+|u_s^m|_H)^{p_0-2}$. Hence,

$$\begin{aligned} I &\le C\Big[\mathbb{E}\int_0^T f_s^{\frac{p_0}{2}} ds + T + \mathbb{E}\int_0^T |u_s^m|_H^{p_0} ds + \mathbb{E}\int_0^T |u_s^m|_V^\alpha ds \\ &\qquad + \mathbb{E}\int_0^T |u_s^m|_V^\alpha (1+|u_s^m|_H)^{p_0-2} ds\Big] \qquad (2.30) \\ &\le C\Big[\mathbb{E}\int_0^T f_s^{\frac{p_0}{2}} ds + T + T\sup_{0\le s\le T}\mathbb{E}|u_s^m|_H^{p_0} + \mathbb{E}\int_0^T |u_s^m|_V^\alpha ds + \int_0^T |u_s^m|_V^\alpha |u_s^m|_H^{p_0-2} ds\Big]\,. \end{aligned}$$

By using (2.26) in (2.30), we obtain (2.28). Furthermore, by Remark 2.1, we get

$$\begin{aligned} &\mathbb{E}\int_0^T \sum_{j=1}^\infty |B_s^j(u_s^m)|_H^2 ds \\ &\le C\Big[T + \mathbb{E}\int_0^T f_t^{\frac{p_0}{2}} ds + \mathbb{E}\int_0^T |u_s^m|_H^{p_0} ds + \mathbb{E}\int_0^T |u_s^m|_V^\alpha ds + \mathbb{E}\int_0^T |u_s^m|_V^\alpha (1+|u_s^m|_H)^{p_0-2} ds\Big] \\ &\le C\Big[T + \mathbb{E}\int_0^T f_t^{\frac{p_0}{2}} ds + T\sup_{s\in[0,T]}\mathbb{E}|u_s^m|_H^{p_0} + \mathbb{E}\int_0^T |u_s^m|_V^\alpha ds + \mathbb{E}\int_0^T |u_s^m|_V^\alpha |u_s^m|_H^{p_0-2} ds\Big] \end{aligned}$$

and hence by using (2.26), we get (2.29). □

2.4 Existence of solution

Having obtained the necessary a priori estimates, we now extract weakly convergent subsequences using the compactness argument. After that we use the local monotonicity condition to establish the existence of a solution to (2.1).

Lemma 2.2. *Let Assumptions A-2.2, A-2.3, A-2.4 and* (2.24) *hold. Then there is a subsequence* $(m_k)_{k\in\mathbb{N}}$ *and* *i) there exists a process* $u \in L^\alpha((0,T)\times\Omega;V)$ *such that*

$$u^{m_k} \rightharpoonup u \;\; in \;\; L^\alpha((0,T)\times\Omega;V),$$

ii) there exists a process $a \in L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega;V^*)$ *such that*

$$A(u^{m_k}) \rightharpoonup a \;\; in \;\; L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega;V^*),$$

iii) there exists a process $b \in L^2((0,T)\times\Omega;\ell^2(H))$ *such that*

$$B(u^{m_k}) \rightharpoonup b \;\; in \;\; L^2((0,T)\times\Omega;\ell^2(H)).$$

Proof. The Banach spaces $L^\alpha((0,T)\times\Omega;V)$, $L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega;V^*)$ and $L^2((0,T)\times\Omega;\ell^2(H))$ are reflexive. Thus, due to Lemma 2.1, there exists a subsequence m_k (see, e.g., Theorem 3.18 in [2]) such that

(i) $u^{m_k} \rightharpoonup v$ in $L^\alpha((0,T)\times\Omega;V)$

(ii) $A(u^{m_k}) \rightharpoonup a$ in $L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega;V^*)$

(iii) $(B^j(u^{m_k}))_{j=1}^{m_k} \rightharpoonup (b^j)_{j=1}^\infty$ in $L^2((0,T)\times\Omega;\ell^2(H))$

as desired. □

Whilst not needed to prove here, it is also possible to show that there is a subsequence of (m_k), again denoted by m_k such that u^{m_k} converges weakly star to u in $L^p(\Omega;L^\infty(0,T;H))$. This is a consequence of Lemma 2.1 and Lemma 1.10.

Lemma 2.3. *Let Assumptions A-2.2, A-2.3 and A-2.4 together with* (2.24) *hold. Then,*

i) for $dt\times\mathbb{P}$ *almost everywhere,*

$$u_t = u_0 + \int_0^t a_s ds + \sum_{j=1}^\infty \int_0^t b_s^j dW_s^j$$

and moreover almost surely $u \in C([0,T];H)$ *and for all* t,

$$|u_t|_H^2 = |u_0|_H^2 + \int_0^t \Big[2\langle a_s, u_s\rangle + \sum_{j=1}^\infty |b_s^j|_H^2\Big] ds + 2\sum_{j=1}^\infty \int_0^t (u_s, b_s^j) dW_s^j\,. \tag{2.31}$$

ii) Finally, $u \in L^2(\Omega;C([0,T];H))$.

Proof. Using Itô's isometry, it can be shown that the stochastic integral is a bounded linear operator from $L^2((0,T)\times\Omega;\ell^2(H))$ to $L^2((0,T)\times\Omega;H)$ and hence maps a weakly convergent sequence to a weakly convergent sequence. Thus, we obtain

$$\sum_{j=1}^{m_k}\int_0^\cdot B_s^j(u_s^{m_k})dW_s^j \rightharpoonup \sum_{j=1}^\infty \int_0^\cdot b_s^j dW_s^j$$

in $L^2((0,T)\times\Omega;H)$, i.e. for any $\psi \in L^2((0,T)\times\Omega;H)$,

$$\mathbb{E}\int_0^T \Big(\sum_{j=1}^{m_k}\int_0^t B_s^j(u_s^{m_k})dW_s^j, \psi(t)\Big)dt \to \mathbb{E}\int_0^T \Big(\sum_{j=1}^\infty \int_0^t b_s^j dW_s^j, \psi(t)\Big)dt. \tag{2.32}$$

Similarly, using Holder's inequality it can be shown that the Bochner integral is a bounded linear operator from $L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega;V^*)$ to $L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega;V^*)$ and is thus continuous with respect to weak topologies. Therefore, for any $\psi \in L^\alpha((0,T)\times\Omega;V)$,

$$\mathbb{E}\int_0^T \Big\langle \int_0^t A_s(u_s^{m_k})ds, \psi(t)\Big\rangle dt \to \mathbb{E}\int_0^T \Big\langle \int_0^t a_s ds, \psi(t)\Big\rangle dt. \tag{2.33}$$

Fix $n \in \mathbb{N}$. Then for any $\phi \in V_n$ and an adapted real valued process η_t bounded by a constant C, we have, for any $k \geq n$,

$$\mathbb{E}\int_0^T \eta_t(u_t^{m_k}, \phi)\, dt = \mathbb{E}\int_0^T \eta_t\Big((u_0^{m_k}, \phi) + \int_0^t \langle A_s(u_s^{m_k}), \phi\rangle\, ds + \sum_{j=1}^{m_k}\int_0^t (\phi, B_s^j(u_s^{m_k}))dW_s^j\Big)\, dt.$$

Taking the limit $k \to \infty$ and using (2.24), (2.32) and (2.33), we obtain

$$\mathbb{E}\int_0^T \eta_t(v_t, \phi)\, dt = \mathbb{E}\int_0^T \eta_t\Big((u_0, \phi) + \int_0^t \langle a_s, \phi\rangle\, ds + \sum_{j=1}^{\infty}\int_0^t (\phi, b_s^j)\, dW_s^j\Big)\, dt$$

with any $\phi \in V_n$ and any adapted real valued process η_t bounded by a constant C. Since $\cup_{n\in\mathbb{N}} V_n$ is dense in V, we obtain

$$v_t = u_0 + \int_0^t a_s ds + \sum_{j=1}^{\infty}\int_0^t b_s^j dW_s^j \tag{2.34}$$

$dt \times \mathbb{P}$ almost everywhere. Now, using Theorem 3.2 on Itô's formula from [28], there exists an H-valued continuous modification u of v which is equal to the right hand side of (2.34) almost surely for all $t \in [0, T]$. Moreover (2.31) holds almost surely for all $t \in [0, T]$. This completes the proof of part (i) of the lemma.

It remains to prove part (ii) of the lemma. To that end, consider the sequence of stopping times σ_n defined by (2.8). From the Burkholder–Davis–Gundy inequality along with Cauchy–Schwarz's and Young's inequalities, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}(u_s, b_s^j)dW_s^j\Big| &\leq 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|(u_s, b_s^j)|_H^2 ds\Big)^{\frac{1}{2}} \\
&\leq 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|u_s|_H^2|b_s^j|_H^2 ds\Big)^{\frac{1}{2}} \\
&\leq 4\mathbb{E}\Big(\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|b_s^j|_H^2 ds\Big)^{\frac{1}{2}} \\
&\leq \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 + C\mathbb{E}\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|b_s^j|_H^2 ds.
\end{aligned} \tag{2.35}$$

Replacing t by $t \wedge \sigma_n$, taking supremum and then expectation in (2.31) and using Hölder's inequality along with (2.35), we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 \leq \mathbb{E}|u_0|_H^2 &+ 2\Big(\mathbb{E}\int_0^T |a_s|^{\frac{\alpha}{\alpha-1}} ds\Big)^{\frac{\alpha-1}{\alpha}}\Big(\mathbb{E}\int_0^T |u_s|_V^\alpha ds\Big)^{\frac{1}{\alpha}} \\
&+ \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 + C\mathbb{E}\sum_{j=1}^{\infty}\int_0^T |b_s^j|_H^2 ds
\end{aligned}$$

which on choosing ϵ small enough gives

$$\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 \leq C\Big[\mathbb{E}|u_0|_H^2 + \Big(\mathbb{E}\int_0^T |a_s|^{\frac{\alpha}{\alpha-1}} ds\Big)^{\frac{\alpha-1}{\alpha}}\Big(\mathbb{E}\int_0^T |u_s|_V^\alpha ds\Big)^{\frac{1}{\alpha}} + \mathbb{E}\sum_{j=1}^{\infty}\int_0^T |b_s^j|_H^2 ds\Big].$$

Finally taking $n \to \infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|u_t|_H^2 < \infty\,.$$

This concludes the proof. $\square$

From now onwards, the processes v and u will be denoted by u for notational convenience. In order to prove that the process u is the solution of equation (2.1), it remains to show that $dt \times \mathbb{P}$ almost everywhere $A(v) = a$ and $B^j(v) = b^j$ for all $j \in \mathbb{N}$. Recall that Ψ and ρ were given in Definition 2.2.

Proof of Theorem 2.3. For $\psi \in L^\alpha((0,T)\times\Omega;V)\cap\Psi\cap L^2(\Omega;C([0,T];H))$, using the product rule and Itô's formula we obtain

$$\begin{aligned}
&\mathbb{E}\big(e^{-\int_0^t \rho(\psi_s)ds}|u_t|_H^2\big) - \mathbb{E}(|u_0|_H^2) \\
&\qquad= \mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle a_s, u_s\rangle + \sum_{j=1}^\infty |b_s^j|_H^2 - \rho(\psi_s)|u_s|_H^2\Big)ds\Big]
\end{aligned}\tag{2.36}$$

and

$$\begin{aligned}
&\mathbb{E}\big(e^{-\int_0^t \rho(\psi_s)ds}|u_t^{m_k}|_H^2\big) - \mathbb{E}(|u_0^{m_k}|_H^2) \\
&\qquad= \mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle A_s(u_s^{m_k}), u_s^{m_k}\rangle + \sum_{j=1}^{m_k} |B_s^j(u_s^{m_k})|_H^2 - \rho(\psi_s)|u_s^{m_k}|_H^2\Big)ds\Big]
\end{aligned}$$

for all $t \in [0,T]$. Note that in view of Remark 2.3, all the integrals are well defined in what follows. Moreover,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle A_s(u_s^{m_k}), u_s^{m_k}\rangle + \sum_{j=1}^{m_k} |B_s^j(u_s^{m_k})|_H^2 - \rho(\psi_s)|u_s^{m_k}|_H^2\Big)ds\Big] \\
&=\mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle A_s(u_s^{m_k}) - A_s(\psi_s), u_s^{m_k} - \psi_s\rangle + 2\langle A_s(\psi_s), u_s^{m_k}\rangle \\
&\quad + 2\langle A_s(u_s^{m_k}) - A_s(\psi_s), \psi_s\rangle + \sum_{j=1}^{m_k}\big|B_s^j(u_s^{m_k}) - B_s^j(\psi_s)\big|_H^2 - \sum_{j=1}^{m_k}|B_s^j(\psi_s)|_H^2 \\
&\quad + 2\sum_{j=1}^{m_k}\big(B_s^j(u_s^{m_k}), B_s^j(\psi_s)\big) - \rho(\psi_s)\left[|u_s^{m_k} - \psi_s|_H^2 - |\psi_s|_H^2 + 2(u_s^{m_k}, \psi_s)\right]\Big)ds\Big].
\end{aligned}$$

Now applying the local monotonicity Assumption A-2.2, we see that

$$\begin{aligned}
&\mathbb{E}\big(e^{-\int_0^t \rho(\psi_s)ds}|u_t^{m_k}|_H^2\big) - \mathbb{E}(|u_0^{m_k}|_H^2) \\
&\qquad\le\mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle A_s(\psi_s), u_s^{m_k}\rangle + 2\langle A_s(u_s^{m_k}) - A_s(\psi_s), \psi_s\rangle \\
&\qquad\quad - \sum_{j=1}^{m_k}|B_s^j(\psi_s)|_H^2 + 2\sum_{j=1}^{m_k}\big(B_s^j(u_s^{m_k}), B_s^j(\psi_s)\big) + \rho(\psi_s)\big[|\psi_s|_H^2 - 2(u_s^{m_k}, \psi_s)\big]\Big)ds\Big].
\end{aligned}$$

Integrating over t from 0 to T, letting $k \to \infty$ and using the weak lower semicontinuity of the norm we obtain,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big(e^{-\int_0^t \rho(\psi_s)ds}|u_t|_H^2 - |u_0|_H^2\big)dt\Big] \\
&\quad\le \liminf_{k\to\infty}\mathbb{E}\Big[\int_0^T \big(e^{-\int_0^t \rho(\psi_s)ds}|u_t^{m_k}|_H^2 - |u_0^{m_k}|_H^2\big)dt\Big] \\
&\quad\le\mathbb{E}\Big[\int_0^T\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle A_s(\psi_s), u_s\rangle + 2\langle a_s - A_s(\psi_s), \psi_s\rangle - \sum_{j=1}^\infty|B_s^j(\psi_s)|_H^2 \\
&\qquad + 2\sum_{j=1}^\infty(b_s^j, B_s^j(\psi_s)) + \rho(\psi_s)\left[|\psi_s|_H^2 - 2(u_s, \psi_s)\right]\Big)dsdt\Big].
\end{aligned}\tag{2.37}$$

Integrating from 0 to T in (2.36) and combining this with (2.37) leads to

$$\mathbb{E}\Big[\int_0^T \int_0^t e^{-\int_0^s \rho(\psi_r)dr}\Big(2\langle a_s - A_s(\psi_s), u_s - \psi_s\rangle \\ + \sum_{j=1}^{\infty} |B_s^j(\psi_s) - b_s^j|_H^2 - \rho(\psi_s)|u_s - \psi_s|_H^2\Big)dsdt\Big] \leq 0. \tag{2.38}$$

Further, using Definition 2.2, Lemmas 2.2 and 2.3,

$$u \in L^\alpha((0,T)\times\Omega; V) \cap \Psi \cap L^2(\Omega; C([0,T]; H)).$$

Taking $\psi = u$ in (2.38), we obtain that $B(u) = b$ in $L^2((0,T)\times\Omega; \ell^2(H))$. Let $\eta \in L^\infty((0,T)\times\Omega;\mathbb{R})$, $\phi \in V$, $\epsilon \in (0,1)$ and let $\psi = u - \epsilon\eta\phi$. Then from (2.38) we obtain that

$$\mathbb{E}\Big[\int_0^T\int_0^t e^{-\int_0^s \rho(u_r - \epsilon\eta_r\phi)dr}\Big(2\epsilon\langle a_s - A_s(u_s - \epsilon\eta_s\phi), \eta_s\phi\rangle - \epsilon^2\rho(u_s - \epsilon\eta_s\phi)|\eta_s\phi|_H^2\Big)dsdt\Big] \leq 0.$$

Dividing by ϵ, letting $\epsilon \to 0$, using Lebesgue dominated convergence theorem and Assumption A-2.1 leads to

$$\mathbb{E}\Big[\int_0^T \int_0^t e^{-\int_0^s \rho(u_r)dr} 2\eta_s\langle a_s - A_s(u_s), \phi\rangle dsdt\Big] \leq 0.$$

Since this holds for any $\eta \in L^\infty((0,T)\times\Omega;\mathbb{R})$ and $\phi \in V$, we get that $A(u) = a$ in $L^{\frac{\alpha}{\alpha-1}}((0,T)\times\Omega; V^*)$, which concludes the proof. $\square$

2.5 Examples

In this section, some examples of stochastic evolution equations are presented which fit in the framework of this chapter and yet do not satisfy the assumptions of [28, 30].

Throughout the section, $\mathscr{D} \subseteq \mathbb{R}^d$ denotes an open bounded domain with smooth boundary.

Example 2.1 (Semi-linear equation)**.** Let γ be a constant such that $\gamma^2 < \frac{1}{3}$. For $i = 1, 2, \ldots, d$, let $g_i : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz continuous and $h_i : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|f(x)| \leq K(1 + |x|^3) \ \text{ and } \ (f(x) - f(y))(x - y) \leq K(1 + |y|^2)|x - y|^2 \ \ \forall x, y \in \mathbb{R}.$$

Consider the stochastic partial differential equation

$$du_t = \big(\Delta u_t + g(u_t)\,\nabla u_t + f(u_t)\big)\,dt + \big(\gamma\nabla u_t + h(u_t)\big)\,dW_t \ \text{ on } \ (0,T)\times\mathscr{D}, \tag{2.39}$$

where $u_t = 0$ on $\partial\mathscr{D}$, u_0 is a given $\mathcal{F}_0$-measurable random variable and Δ is the Laplace operator defined in (1.3). Moreover W is an $\mathbb{R}^d$-valued Wiener process. It will now be shown that such an equation, in its weak form, satisfies Assumptions A-2.1 to A-2.4.

Let $A : W_0^{1,2}(\mathscr{D}) \to W^{-1,2}(\mathscr{D})$ and $B^i : W_0^{1,2}(\mathscr{D}) \to L^2(\mathscr{D})$ be given by

$$A(u) := \Delta u + g(u)\nabla u + f(u) \text{ and } B^i(u) := \gamma D_i u + h_i(u)$$

for $i = 1, 2, \ldots d$. The next step is to show that these operators satisfy Assumptions A-2.1 to A-2.4. We immediately notice that A-2.1 holds, in particular, since g and f are continuous.

We now wish to verify the local monotonicity condition. By using the assumptions imposed on f and g we see for $u, v \in W_0^{1,2}(\mathscr{D})$, upon application of Hölder's inequality, that

$$\begin{aligned}
&\langle A(u) - A(v), u - v\rangle \\
&= -|u - v|_{W_0^{1,2}}^2 + \langle g(u)\nabla(u - v), u - v\rangle + \langle(\nabla v)(g(u) - g(v)), u - v\rangle + \langle f(u) - f(v), u - v\rangle \\
&\leq -|u - v|_{W_0^{1,2}}^2 + C|u - v|_{W_0^{1,2}}|u - v|_{L^2} + C|v|_{W_0^{1,2}}|u - v|_{L^4}^2 + C|u - v|_{L^2}^2 + C|v|_{L^4}^2|u - v|_{L^4}^2 .
\end{aligned}$$

Then (1.7) implies that

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle \leq &-|u-v|^2_{W_0^{1,2}}+C|u-v|_{W_0^{1,2}}|u-v|_{L^2}+C|v|_{W_0^{1,2}}|u-v|_{L^2}|u-v|_{W_0^{1,2}}\\ &+C|u-v|^2_{L^2}+C|v|^2_{L^4}|u-v|_{W_0^{1,2}}|u-v|_{L^2}.\end{aligned}$$

Young's inequality with some $\epsilon>0$ finally leads to

$$\langle A(u)-A(v),u-v\rangle \leq (\epsilon-1)|u-v|^2_{W_0^{1,2}}+C(1+|v|^2_{W_0^{1,2}}+|v|^4_{L^4})|u-v|^2_{L^2}. \tag{2.40}$$

Moreover,

$$\sum_{i=1}^d |B^i(u)-B^i(v)|^2_{L^2}\leq 2\gamma^2|u-v|^2_{W_0^{1,2}}+C|u-v|^2_{L^2}\,.$$

Thus using (1.7) once again, we obtain

$$\begin{aligned}2\langle A(u)-A(v),u-v\rangle+&\sum_{i=1}^d |B^i(u)-B^i(v)|^2_{L^2}\\ &\leq (2\epsilon+2\gamma^2-2)|u-v|^2_{W_0^{1,2}}+C(1+|v|^2_{W_0^{1,2}}+|v|^2_{L^2}|v|^2_{W_0^{1,2}})|u-v|^2_{L^2}\,.\end{aligned}$$

If $\gamma\in(-1,1)$, then one can get that for some $\theta>0$,

$$\begin{aligned}2\langle A(u)-A(v),u-v\rangle+\sum_{i=1}^d |B^i(u)-B^i(v)|^2_{L^2}+\theta|u-v|^2_{W_0^{1,2}}&\\ \leq C(1+|v|^2_{W_0^{1,2}})(1+|v|^2_{L^2})|u-v|^2_{L^2},&\end{aligned}$$

for all $u,v\in W_0^{1,2}(\mathscr{D})$. Hence Assumption A-2.2 is satisfied with $\alpha:=2$ and $\beta:=2$.

The next condition that ought to be verified is p_0-stochastic coercivity. Taking $v=0$ in (2.40) and observing that $A(0)=f(0)$ is a constant, on using Hölder's inequality we obtain for all $u\in W_0^{1,2}(\mathscr{D})$

$$\langle A(u),u\rangle\leq(\epsilon-1)|u|^2_{W_0^{1,2}}+C|u|^2_{L^2}+\langle f(0),u\rangle\leq(\epsilon-1)|u|^2_{W_0^{1,2}}+C|u|^2_{L^2}$$

which implies, together with the assumptions on h, that

$$2\langle A(u),u\rangle+(p_0-1)\sum_{i=1}^d|B^i(u)|^2_{L^2}\leq\big(2\epsilon+2\gamma^2(p_0-1)-2\big)\,|u|^2_{W_0^{1,2}}+C\big(1+|u|^2_{L^2}\big).$$

Taking[6] $p_0:=4$ we see that if $\gamma^2<1/3$, then Assumption A-2.3 holds with $\theta:=2-2\epsilon-6\gamma^2$ for $\epsilon>0$ sufficiently small.

Finally we wish to verify the growth condition. Using the boundedness of g and Hölder's inequality, we obtain

$$|g(u)\nabla u|_{W^{-1,2}}\leq C|u|_{W_0^{1,2}}\quad\forall u\in W_0^{1,2}(\mathscr{D}).$$

Moreover, due to Hölder's inequality, we get that for any $1\leq q<\infty$ and $u,v\in W_0^{1,2}(\mathscr{D})$,

$$\langle f(u),v\rangle\leq C|v|_{L^q}+C|u|^3_{L^{\frac{3q}{q-1}}}|v|_{L^q}\leq C|v|_{W_0^{1,2}}+C|u|^3_{L^{\frac{3q}{q-1}}}|v|_{W_0^{1,2}},$$

where the last inequality is a consequence of the Sobolev embedding and the fact that $d=1$ or 2. Hence, with $q=6$ we obtain

$$|f(u)|_{W^{-1,2}}\leq C\left(1+|u|^3_{L^{\frac{18}{5}}}\right)\leq C\left(1+|u|_{L^2}|u|^2_{L^6}\right),$$

where the last inequality follows from interpolation between spaces of integrable functions, see

[6]One may choose $p_0\geq\beta+2$ but clearly that will put more restriction on γ.

e.g. [42, Theorem 1.24]. Finally, using the Sobolev embedding again, we see that

$$|A(u)|^2_{W^{-1,2}} \leq C\left(1 + |u|^2_{W^{1,2}_0}\right)\left(1 + |u|^2_{L^2}\right)$$

thus Assumption A-2.4 is satisfied with $\alpha = 2$, $\beta = 2$.

If $d = 1$ or 2, $\alpha = 2$, $\beta = 2$, $p_0 = 4$, $\gamma^2 < 1/3$ and $u_0 \in L^4(\Omega; L^2(\mathscr{D}))$ is $\mathscr{F}_0$-measurable then, in view of Theorems 2.1, 2.2 and 2.3, we conclude that equation (2.39) has a unique solution and moreover for any $p < 4$ we have

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |u_t|^p_{L^2} + \int_0^T |u_t|^2_{W^{1,2}_0} dt\Big) < C\left(1 + \mathbb{E}|u_0|^4_{L^2}\right).$$

Example 2.2 (Stochastic Burgers equation)**.** Let $d = 1$ and $\mathscr{D} = (0,1)$. Let $\gamma \in (-\sqrt{1/3}, \sqrt{1/3})$ be a constant and let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Consider the stochastic partial differential equation

$$du_t = \Big(\Delta u_t + u_t D u_t\Big)dt + \big(\gamma D u_t + h(u_t)\big)dW_t \ \text{ on } (0,T)\times\mathscr{D}, \tag{2.41}$$

where $u_t = 0$ on $\partial\mathscr{D}$ and an $L^2(\mathscr{D})$-valued, $\mathcal{F}_0$-measurable u_0 is a given initial condition. Here W is a real-valued Wiener process. Weak formulation of this equation can be interpreted as a stochastic evolution equation as follows.

Define $A : W^{1,2}_0(\mathscr{D}) \to W^{-1,2}(\mathscr{D})$ and $B : W^{1,2}_0(\mathscr{D}) \to L^2(\mathscr{D})$ as

$$A(u) := \Delta u + uDu \quad \text{and} \quad B(u) := \gamma D u + h(u).$$

Note that SPDE (2.41) is not covered by Example 2.1 as the function $g(u) = u$ is not bounded. However, Assumption A-2.1 is satisfied following the same arguments as in Example 2.1. Next, we would like to check the local monotonicity assumption. Note that, if $u, v \in W^{1,2}_0(\mathscr{D})$, then

$$\frac{1}{2}D\left[u^2 - v^2\right] = uDu - vDv$$

and so using integration by parts,

$$\langle uDu - vDv, u - v\rangle = -\frac{1}{2}\big\langle u^2 - v^2, D(u-v)\big\rangle.$$

Thus,

$$\big\langle A(u) - A(v), u - v\big\rangle = -|u-v|^2_{W^{1,2}_0} - \frac{1}{2}\big\langle (u-v)^2, D(u-v)\big\rangle - \big\langle v(u-v), D(u-v)\big\rangle.$$

But using integration by parts again we see that $\big\langle (u-v)^2, D(u-v)\big\rangle = 0$ and so

$$\langle A(u) - A(v), u - v\rangle = -|u-v|^2_{W^{1,2}_0} - \big\langle v(u-v), D(u-v)\big\rangle.$$

Further, using Hölder's inequality we get

$$\langle A(u) - A(v), u - v\rangle \leq -|u-v|^2_{W^{1,2}_0} + |v|_{L^4}|u-v|_{L^4}|u-v|_{W^{1,2}_0}$$

and thus Gagliardo–Nirenberg inequality, and Young's inequality imply that for any $\epsilon > 0$,

$$\langle A(u) - A(v), u - v\rangle \leq -|u-v|^2_{W^{1,2}_0} + \epsilon|u-v|^2_{W^{1,2}_0} + C|v|^2_{L^2}|v|^2_{W^{1,2}_0}|u-v|^2_{L^2}. \tag{2.42}$$

This, along with Lipschitz continuity of h, gives

$$\begin{aligned} 2\langle A(u) - A(v), u - v\rangle + &|B(u) - B(v)|^2_{L^2} \\ &\leq (-2 + 2\epsilon + 2\gamma^2)|u-v|^2_{W^{1,2}_0} + C(1 + |v|^2_{L^2})(1 + |v|^2_{W^{1,2}_0})|u-v|^2_{L^2} \end{aligned}$$

for all $u, v \in W_0^{1,2}(\mathscr{D})$. As $\gamma^2 \in (0, 1/3)$, we can choose $\epsilon > 0$ sufficiently small so that $-1+\epsilon+\gamma^2 < 0$ and hence Assumption A-2.2 is satisfied with $\alpha := 2$ and $\beta := 2$. The next step is to show that the p_0-stochastic coercivity assumption holds with $p_0 = 4$. Indeed, substituting $v = 0$ in (2.42), we obtain

$$\langle A(u), u\rangle \le (-1+\epsilon)|u|^2_{W_0^{1,2}}$$

which along with linear growth of h implies that

$$2\langle A(u), u\rangle + 3|B(u)|^2_{L^2} \le (-2+2\epsilon+6\gamma^2)|u|^2_{W_0^{1,2}} + C\big(1+|u|^2_{L^2}\big).$$

Note that since $\gamma^2 \in (0, 1/3)$ we can choose $\epsilon > 0$ sufficiently small so that $\theta := 2-2\epsilon-6\gamma^2 > 0$. Then with $f := C$, Assumption A-2.3 holds.

Finally, we verify the growth assumption on A. Using integration by parts, Hölder's inequality and Gagliardo–Nirenberg inequality, we obtain for $u, v \in W_0^{1,2}(\mathscr{D})$,

$$\begin{aligned}\langle uDu, v\rangle &= -\frac{1}{2}\langle u^2, Dv\rangle \le \frac{1}{2}|u|^2_{L^4}|v|_{W_0^{1,2}}\\ &\le C|u|_{L^2}|u|_{W_0^{1,2}}|v|_{W_0^{1,2}}\end{aligned}$$

which then implies that

$$|uDu|_{W^{-1,2}} \le C|u|_{L^2}|u|_{W_0^{1,2}}. \tag{2.43}$$

Hence using (1.4), we obtain for all $u \in W_0^{1,2}(\mathscr{D})$

$$|A(u)|^2_{W^{-1,2}} \le C|u|^2_{W_0^{1,2}}(1+|u|^2_{L^2})$$

proving that Assumption A-2.4 is satisfied for $\alpha = 2, \beta = 2$ and $f = C$. Thus, in view of Theorems 2.1, 2.2 and 2.3, if $u_0 \in L^4(\Omega; L^2(\mathscr{D}))$, then SPDE (2.41) has a unique solution $(u_t)_{t\in[0,T]}$ and for any $p < 4$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|u_t|^p_{L^2} + \int_0^T |u_t|^2_{W_0^{1,2}}dt\Big) < C\left(1+\mathbb{E}|u_0|^4_{L^2}\right),$$

where we recall in particular that C depends on T.

Example 2.3. Let $d = 1$ and $\mathscr{D} = (0,1)$. Let $\gamma \in (-\sqrt{2/5}, \sqrt{2/5})$ be a constant. Consider the stochastic partial differential equation

$$du_t = \Big(\Delta u_t + u_t Du_t - u_t^3\Big)dt + \gamma u_t^2 dW_t \ \text{ on } (0,T)\times\mathscr{D}, \tag{2.44}$$

where $u_t = 0$ on $\partial\mathscr{D}$ and an $L^2(\mathscr{D})$-valued, $\mathcal{F}_0$-measurable u_0 is a given initial condition. Here W is a real-valued Wiener process. Weak formulation of this equation can be interpreted as a stochastic evolution equation as follows.

Define $A : W_0^{1,2}(\mathscr{D}) \to W^{-1,2}(\mathscr{D})$ and $B : W_0^{1,2}(\mathscr{D}) \to L^2(\mathscr{D})$ as

$$A(u) := \Delta u + uDu - u^3 \quad\text{and}\quad B(u) := \gamma u^2.$$

where, A and B are well-defined using the Sobolev embedding $W_0^{1,2}(\mathscr{D}) \subset L^\infty(\mathscr{D})$ and the Gagliardo–Nirenberg inequality (1.6). Clearly, Assumption A-2.1 is satisfied. Further, using Mean value theorem it is easy to observe that

$$\langle -u^3+v^3, u-v\rangle + |\gamma(u^2-v^2)|^2_{L^2} \le 0$$

since $\gamma^2 < 2/5$ and hence using (2.42), we obtain

$$2\langle A(u)-A(v), u-v\rangle + |B(u)-B(v)|^2_{L^2}$$

$$\le (-2+2\epsilon)|u-v|^2_{W^{1,2}_0} + C(1+|v|^2_{L^2})(1+|v|^2_{W^{1,2}_0})|u-v|^2_{L^2}$$
$$\le (-2+2\epsilon)|u-v|^2_{W^{1,2}_0} + C(1+|v|^4_{L^2})(1+|v|^2_{W^{1,2}_0})|u-v|^2_{L^2}$$

for any $\epsilon > 0$ and for all $u, v \in W^{1,2}_0(\mathscr{D})$. By choosing $0 < \epsilon < 1$, Assumption A-2.2 is satisfied with $\alpha :=\ 2$ and $\beta := 4$.

Further Assumption A-2.3 holds with $p_0 = 6$ and $\theta = 2 - 2\epsilon$. Indeed, we have

$$2\langle A(u), u\rangle + 5|B(u)|^2_{L^2} \le (-2+2\epsilon)|u|^2_{W^{1,2}_0}.$$

Finally, we verify the growth assumption on A. Using Sobolev embedding we obtain for $u, v \in\ W^{1,2}_0(\mathscr{D})$,

$$|\langle -u^3, v\rangle| \le |u|_\infty |v|_\infty |u|^2_{L^2} \le C|u|_{W^{1,2}_0} |v|_{W^{1,2}_0} |u|^2_{L^2}$$

which then implies that

$$|-u^3|_{W^{-1,2}} \le C|u|^2_{L^2}|u|_{W^{1,2}_0}.$$

Hence using (1.4) and (2.43) we obtain for all $u \in W^{1,2}_0(\mathscr{D})$,

$$|A(u)|^2_{W^{-1,2}} \le C|u|^2_{W^{1,2}_0}(1+|u|^4_{L^2})$$

proving that Assumption A-2.4 is satisfied for $\alpha = 2, \beta = 4$ and $f = C$.

Thus, in view of Theorems 2.1, 2.2 and 2.3, if $u_0 \in L^6(\Omega; L^2(\mathscr{D}))$, then equation (2.44) has a unique solution $(u_t)_{t\in[0,T]}$ and for any $p < 6$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |u_t|^p_{L^2} + \int_0^T |u_t|^2_{W^{1,2}_0} dt\Big) < C\left(1 + \mathbb{E}|u_0|^6_{L^2}\right).$$

Example 2.4 (Stochastic p-Laplace equation)**.** For $\alpha > 2$ and γ to be chosen suitably, consider the stochastic partial differential equation

$$du_t = \Big(\sum_{i=1}^d D_i\big(|D_i u_t|^{\alpha-2} D_i u_t\big) + f(u_t)\Big)\, dt + \sum_{i=1}^d \gamma |D_i u_t|^{\frac{\alpha}{2}}\, dW^i_t + \sum_{i\in\mathbb{N}} h_i(u_t)\, dW^i_t \tag{2.45}$$

on $(0,T)\times\mathscr{D}$, where $u_t = 0$ on $\partial\mathscr{D}$ and u_0 is a given $\mathcal{F}_0$-measurable random variable. Moreover W^i are independent Wiener processes. Further, assume that there are constants $r, s, t \ge 1$ and continuous function f on $\mathbb{R}$ such that

$$\begin{aligned} & f(x)x \le K(1+|x|^{\frac{\alpha}{2}+1}); \\ & |f(x)| \le K(1+|x|^r) \\ \text{and}\ \ & (f(x)-f(y))(x-y) \le K(1+|y|^s)|x-y|^t \ \ \forall x, y \in \mathbb{R}. \end{aligned}$$

Finally, for $i \in \mathbb{N}$, let $h_i : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constants M_i such that the sequence $(M_i)_{i\in\mathbb{N}} \in \ell^2$. Let $A : W^{1,\alpha}_0(\mathscr{D}) \to W^{-1,\alpha}(\mathscr{D})$ be given by

$$A(u) := \sum_{i=1}^d D_i\big(|D_i u|^{\alpha-2} D_i u\big) + f(u)$$

and $B^i : W^{1,\alpha}_0(\mathscr{D}) \to L^2(\mathscr{D})$ be given by,

$$B^i(u) := \begin{cases} \gamma |D_i u|^{\frac{\alpha}{2}} + h_i(u) & \text{for } i = 1, 2, \ldots, d, \\ h_i(u) & \text{otherwise}. \end{cases}$$

It will now be shown that these operators satisfy Assumptions A-2.1 to A-2.4 if any of the following holds:

(1.) $d < \alpha$, $r = \alpha + 1$, $s \leq \alpha$, $t = 2$ and $u_0 \in L^6(\Omega; L^2(\mathscr{D}))$,

(2.) $d > \alpha$, $r = \frac{2\alpha}{d} + \alpha - 1$, $s \leq \min\left\{\frac{\alpha^2(t-2)}{(d-\alpha)(\alpha-2)}, \frac{\alpha(\alpha-t)}{(\alpha-2)}\right\}$, $2 < t < \alpha$ and $u_0 \in L^6(\Omega; L^2(\mathscr{D}))$.

Case (1.) We immediately notice that A-2.1 holds since f is continuous. We now wish to verify the local monotonicity condition. Note that for each $i = 1, 2, \ldots, d$, using integration by parts

$$\begin{aligned}
&\left\langle D_i\big(|D_i u|^{\alpha-2} D_i u\big) - D_i\big(|D_i v|^{\alpha-2} D_i v\big), u - v\right\rangle + \left|\gamma|D_i u|^{\frac{\alpha}{2}} - \gamma|D_i v|^{\frac{\alpha}{2}}\right|_{L^2}^2 \\
&= \int_{\mathscr{D}} \Big[- \big(|D_i u(x)|^{\alpha-2} D_i u(x) - |D_i v(x)|^{\alpha-2} D_i v(x)\big)(D_i u(x) - D_i v(x)) \\
&\quad + \gamma^2\big(|D_i u(x)|^{\frac{\alpha}{2}} - |D_i v(x)|^{\frac{\alpha}{2}}\big)^2\Big] dx.
\end{aligned} \tag{2.46}$$

On using mean value theorem with integral form of remainder, we get

$$\begin{aligned}
|D_i u(x)|^{\alpha-2} D_i u(x) - &|D_i v(x)|^{\alpha-2} D_i v(x) \\
&= (\alpha - 1)\int_0^1 |G_i^\zeta(u,v)(x)|^{\alpha-2}(D_i u(x) - D_i v(x)) d\zeta
\end{aligned}$$

and

$$\begin{aligned}
|D_i u(x)|^{\frac{\alpha}{2}} - &|D_i v(x)|^{\frac{\alpha}{2}} \\
&= \frac{\alpha}{2}\int_0^1 |G_i^\zeta(u,v)(x)|^{\frac{\alpha}{2}-2}(G_i^\zeta(u,v)(x))(D_i u(x) - D_i v(x)) d\zeta
\end{aligned}$$

where, $G_i^\zeta(u,v)(x) := \zeta D_i u(x) + (1-\zeta) D_i v(x)$. Substituting these values in (2.46) and using Jensen's inequality, we get

$$\begin{aligned}
&\left\langle D_i\big(|D_i u|^{\alpha-2} D_i u\big) - D_i\big(|D_i v|^{\alpha-2} D_i v\big), u - v\right\rangle + \left|\gamma|D_i u|^{\frac{\alpha}{2}} - \gamma|D_i v|^{\frac{\alpha}{2}}\right|_{L^2}^2 \\
&= \int_{\mathscr{D}} \Big[\int_0^1 -(\alpha-1)|G_i^\zeta(u,v)(x)|^{\alpha-2}(D_i u(x) - D_i v(x))^2 d\zeta \\
&\quad + \gamma^2 \frac{\alpha^2}{4}\Big(\int_0^1 |G_i^\zeta(u,v)(x)|^{\frac{\alpha}{2}-2}(G_i^\zeta(u,v)(x))(D_i u(x) - D_i v(x)) d\zeta\Big)^2\Big] dx \\
&\leq \int_{\mathscr{D}}\int_0^1 |G_i^\zeta(u,v)(x)|^{\alpha-2}\Big[-(\alpha-1) + \gamma^2\frac{\alpha^2}{4}\Big](D_i u(x) - D_i v(x))^2 d\zeta dx \\
&\leq 0
\end{aligned} \tag{2.47}$$

provided $\gamma^2 \leq \frac{4(\alpha-1)}{\alpha^2}$. Further for $d < \alpha$, by the Sobolev embedding we have $W_0^{1,\alpha}(\mathscr{D}) \subset L^\infty(\mathscr{D})$ and taking $t = 2$ in the assumption imposed on f, we obtain that for $u, v \in W_0^{1,\alpha}(\mathscr{D})$

$$\begin{aligned}
\langle f(u) - f(v), u - v\rangle &\leq K\int_{\mathscr{D}} (1 + |v(x)|^s)|u(x) - v(x)|^2 dx \\
&\leq K(1 + |v|_{L^\infty}^s)|u - v|_{L^2}^2 \\
&\leq C(1 + |v|_{W_0^{1,\alpha}}^s)|u - v|_{L^2}^2 \\
&\leq C(1 + |v|_{W_0^{1,\alpha}}^\alpha)|u - v|_{L^2}^2,
\end{aligned} \tag{2.48}$$

if $s \leq \alpha$. Moreover, using Lipschitz continuity of the functions h_i

$$|h^i(u) - h^i(v)|_{L^2}^2 \leq M_i^2 |u - v|_{L^2}^2, \tag{2.49}$$

where the sequence $(M_i)_{i\in\mathbb{N}} \in \ell^2$.

From (2.47), (2.48) and (2.49), we get for all $u, v \in W_0^{1,\alpha}(\mathscr{D})$

$$
\begin{aligned}
&2\langle A(u)-A(v), u-v\rangle + \sum_{i\in\mathbb{N}} |B^i(u)-B^i(v)|_{L^2}^2 \\
&\quad \le 2\sum_{i=1}^d \Big[\big\langle D_i\big(|D_iu|^{\alpha-2}D_iu\big) - D_i\big(|D_iv|^{\alpha-2}D_iv\big), u-v\big\rangle + \big|\gamma|D_iu|^{\frac{\alpha}{2}} - \gamma|D_iv|^{\frac{\alpha}{2}}\big|_{L^2}^2\Big] \\
&\qquad + 2\Big[\langle f(u)-f(v), u-v\rangle + \sum_{i\in\mathbb{N}} |h^i(u)-h^i(v)|_{L^2}^2\Big] \\
&\quad \le C\Big(1+|v|_{W_0^{1,\alpha}}^\alpha\Big)|u-v|_{L^2}^2 \,.
\end{aligned}
$$

Hence Assumption A-2.2 is satisfied with $\beta := 0$. Again, using integration by parts

$$
2\sum_{i=1}^d \big\langle D_i\big(|D_iu|^{\alpha-2}D_iu\big), u\big\rangle = -2|u|_{W_0^{1,\alpha}}^\alpha \,. \tag{2.50}
$$

Using assumptions on f, Holder's inequality and Sobolev embedding as above, we obtain

$$
\begin{aligned}
2\langle f(u), u\rangle &\le 2K\int_{\mathscr{D}} (1+|u(x)|^{\frac{\alpha}{2}+1})dx \le 2K(1+|u|_{L^\infty}^{\frac{\alpha}{2}}|u|_{L^2}) \\
&\qquad\qquad \le C(1+|u|_{W_0^{1,\alpha}}^{\frac{\alpha}{2}}|u|_{L^2}) \le \delta|u|_{W_0^{1,\alpha}}^\alpha + C(1+|u|_{L^2}^2)
\end{aligned}
$$

where last inequality is obtained using Young's inequality with sufficiently small $\delta > 0$. Further, for any $p_0 > 2$

$$
(p_0-1)\sum_{i=1}^d 2\big|\gamma|D_iu|^{\frac{\alpha}{2}}\big|_{L^2}^2 = (p_0-1)2\gamma^2\sum_{i=1}^d \int_{\mathscr{D}} |D_iu(x)|^\alpha dx = (p_0-1)2\gamma^2|u|_{W_0^{1,\alpha}}^\alpha \,. \tag{2.51}
$$

Hence Assumptions A-2.3 is satisfied with $\theta := 2 - 2(p_0-1)\gamma^2 - \delta > 0$.

Note that using Hölder's inequality, we get for $u, v \in W_0^{1,\alpha}(\mathscr{D})$,

$$
\begin{aligned}
\int_{\mathscr{D}} |D_iu(x)|^{\alpha-1}|D_iv(x)|dx &\le \Big(\int_{\mathscr{D}} |D_iu(x)|^\alpha dx\Big)^{\frac{\alpha-1}{\alpha}}\Big(\int_{\mathscr{D}} |D_iv(x)|^\alpha dx\Big)^{\frac{1}{\alpha}} \\
&\le \Big(\sum_{i=1}^d\int_{\mathscr{D}} |D_iu(x)|^\alpha dx\Big)^{\frac{\alpha-1}{\alpha}}\Big(\sum_{i=1}^d\int_{\mathscr{D}} |D_iv(x)|^\alpha dx\Big)^{\frac{1}{\alpha}} \\
&= |u|_{W_0^{1,\alpha}}^{\alpha-1}|v|_{W_0^{1,\alpha}} \,.
\end{aligned} \tag{2.52}
$$

Further using assumption on f with $r = \alpha+1$ and Sobolev embedding,

$$
\begin{aligned}
\int_{\mathscr{D}} |f(u(x))||v(x)|dx &\le K\int_{\mathscr{D}} \big(1+|u(x)|^{\alpha+1}\big)|v(x)|dx \\
&\le K|v|_{L^\infty}(1+|u|_{L^{\alpha+1}}^{\alpha+1}) \\
&\le K|v|_{W_0^{1,\alpha}}(1+|u|_{L^\infty}^{\alpha-1}|u|_{L^2}^2) \\
&\le K|v|_{W_0^{1,\alpha}}(1+|u|_{W_0^{1,\alpha}}^{\alpha-1}|u|_{L^2}^2)
\end{aligned}
$$

and hence,

$$
\begin{aligned}
|A(u)|_{W^{-1,\alpha}} &\le K|u|_{W_0^{1,\alpha}}^{\alpha-1} + K(1+|u|_{W_0^{1,\alpha}}^{\alpha-1}|u|_{L^2}^2) \\
&\le K(1+|u|_{W_0^{1,\alpha}}^{\alpha-1})(1+|u|_{L^2}^2) \,.
\end{aligned}
$$

Thus, Assumption A-2.4 holds with $\beta = \frac{2\alpha}{\alpha-1} < 4$ and in view of Theorems 2.1, 2.2 and 2.3, we conclude that equation (2.45) has a unique solution and moreover for any $p < \frac{4\alpha-2}{\alpha-1}$ we have,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|u_t|_{L^2}^p + \int_0^T |u_t|^\alpha_{W_0^{1,\alpha}}dt\Big) < C\left(1+\mathbb{E}|u_0|_{L^2}^{\frac{4\alpha-2}{\alpha-1}}\right).$$

Case (2.) In the case $d > \alpha$, we use the Sobolev embedding $W_0^{1,\alpha}(\mathscr{D}) \subset L^{\bar{p}}(\mathscr{D}),\ \bar{p} = \frac{d\alpha}{d-\alpha}$ to verify Assumptions A-2.2 to A-2.4. Let

$$t_0 = \frac{\alpha(t-2)}{t(\alpha-2)} \qquad \text{and} \qquad \frac{1}{p_1} = \frac{1-t_0}{2} + \frac{t_0}{\bar{p}}\ .$$

Then, $t_0 \in (0,1)$ and $p_1 \in (2,\bar{p})$. Thus, we obtain the following interpolation inequality:

$$|u|_{L^{p_1}} \le |u|_{L^2}^{1-t_0}|u|_{L^{\bar{p}}}^{t_0}, \qquad u \in W_0^{1,\alpha}(\mathscr{D}) \tag{2.53}$$

Using the fact $2 < t < \alpha$, we can see that $t < p_1$. Let $p_2 = \frac{p_1}{p_1-t}$, then with some calculations, we have

$$\frac{t}{p_1} = \frac{\alpha-t}{\alpha-2} + \frac{\alpha(t-2)}{\bar{p}(\alpha-2)} \qquad \text{and} \qquad p_2 = \frac{\bar{p}(\alpha-2)}{(\bar{p}-\alpha)(t-2)}.$$

Thus $s \le \frac{\alpha^2(t-2)}{(d-\alpha)(\alpha-2)} = \frac{(\bar{p}-\alpha)(t-2)}{(\alpha-2)}$ implies $sp_2 \le \bar{p}$ and hence we have

$$|u|_{L^{sp_2}} \le C|u|_{L^{\bar{p}}} \le C|u|_{W_0^{1,\alpha}}, \qquad u \in W_0^{1,\alpha}(\mathscr{D}). \tag{2.54}$$

Hence, using assumption on f, Hölder's inequality, interpolation inequality (2.53), Young's inequality, definition of t_0, the fact that $s \le \frac{\alpha(\alpha-t)}{\alpha-2}$ and (2.54), we obtain

$$\begin{aligned}
&\langle f(u)-f(v), u-v\rangle \le K\int_{\mathscr{D}}\big(1+|v(x)|^s\big)|u(x)-v(x)|^t dx \\
&\le C(1+|v|^s_{L^{sp_2}})|u-v|^t_{L^{p_1}} \le C(1+|v|^s_{L^{sp_2}})|u-v|_{L^2}^{t(1-t_0)}|u-v|_{L^{\bar{p}}}^{tt_0} \\
&\le \epsilon|u-v|_{L^{\bar{p}}}^{tt_0\frac{\alpha-2}{t-2}} + C(1+|v|_{L^{sp_2}}^{s\frac{\alpha-2}{\alpha-t}})|u-v|_{L^2}^{t(1-t_0)\frac{\alpha-2}{\alpha-t}} \\
&\le \epsilon|u-v|^\alpha_{L^{\bar{p}}} + C(1+|v|^\alpha_{L^{sp_2}})|u-v|^2_{L^2} \le \epsilon|u-v|^\alpha_{W_0^{1,\alpha}} + C(1+|v|^\alpha_{W_0^{1,\alpha}})|u-v|^2_{L^2}\ .
\end{aligned} \tag{2.55}$$

Further as observed in (2.47), we have

$$\big\langle D_i\big(|D_iu|^{\alpha-2}D_iu\big) - D_i\big(|D_iv|^{\alpha-2}D_iv\big), u-v\big\rangle + 2\big|\gamma|D_iu|^{\frac{\alpha}{2}} - \gamma|D_iv|^{\frac{\alpha}{2}}\big|^2_{L^2} \le 0 \tag{2.56}$$

provided $\gamma^2 \le \frac{2(\alpha-1)}{\alpha^2}$. Moreover, using the inequality

$$(|a|^ra - |b|^rb)(a-b) \ge 2^{-r}|a-b|^{r+2} \quad \forall\ r \ge 0,\ a,b \in \mathbb{R},$$

we have,

$$\big\langle D_i\big(|D_iu|^{\alpha-2}D_iu\big) - D_i\big(|D_iv|^{\alpha-2}D_iv\big), u-v\big\rangle \le -2^{-(\alpha-2)}|D_iu - D_iv|^\alpha_{L^\alpha}\ . \tag{2.57}$$

Thus Assumption A-2.2 follows from (2.49), (2.55), (2.56) and (2.57). Again, using assumption on f, Holder's inequality and Young's inequality, we obtain

$$\begin{aligned}
2\langle f(u), u\rangle &\le K\int_{\mathscr{D}}(1+|u(x)|^{\frac{\alpha}{2}+1})dx \le K(1+|u|_{L^\alpha}^{\frac{\alpha}{2}}|u|_{L^2}) \\
&\le \delta|u|^\alpha_{W_0^{1,\alpha}} + C(1+|u|^2_{L^2})
\end{aligned}$$

and thus using (2.50) and (2.51), Assumption A-2.3 is satisfied with $\theta := 2-2(p_0-1)\gamma^2-\delta$. In order to prove Assumption A-2.4, we observe that for $r = \frac{2\alpha}{d}+\alpha-1$ and the Hölder conjugate

$\bar{p}'$ of $\bar{p}$,

$$\frac{1}{r\bar{p}'} = \frac{\frac{\alpha-1}{r}}{\bar{p}} + \frac{\frac{2\alpha}{dr}}{2}$$

and thus we have the following interpolation inequality:

$$|u|_{L^{r\bar{p}'}} \leq |u|_{L^{\bar{p}}}^{\frac{\alpha-1}{r}} |u|_{L^2}^{\frac{2\alpha}{dr}} .$$

Hence using assumption on f, Hölder's inequality and Sobolev embedding, we get

$$\begin{aligned}\int_{\mathscr{D}} |f(u(x))||v(x)|dx &\leq K \int_{\mathscr{D}} \big(1+|u(x)|^r\big)|v(x)|dx \leq K|v|_{L^{\bar{p}}}\big(1+|u|^r_{L^{r\bar{p}'}}\big) \\ &\leq K|v|_{W^{1,\alpha}_0}\Big(1+|u|^{\alpha-1}_{L^{\bar{p}}}|u|^{\frac{2\alpha}{d}}_{L^2}\big) \leq K|v|_{W^{1,\alpha}_0}(1+|u|^{\alpha-1}_{W^{1,\alpha}_0}|u|^{\frac{2\alpha}{d}}_{L^2}\Big)\end{aligned}$$

and therefore using (2.52), we get

$$|A(u)|_{W^{-1,\alpha}} \leq K|u|^{\alpha-1}_{W^{1,\alpha}_0} + K(1+|u|^{\alpha-1}_{W^{1,\alpha}_0}|u|^{\frac{2\alpha}{d}}_{L^2}) \leq K(1+|u|^{\alpha-1}_{W^{1,\alpha}_0})(1+|u|^{\frac{2\alpha}{d}}_{L^2}) .$$

Thus, Assumption A-2.4 holds with $\beta = 4$ and as in Case (1.), the desired result is obtained.

Remark 2.5. Note that taking $h = 0$ in previous examples, we require $\gamma^2 < 2/3$ in Examples 2.1, 2.2 and less than $\frac{8(\alpha-1)}{\alpha^2} \wedge \frac{2(\alpha-1)}{3\alpha-1}$ in Example 2.4. Here, γ^2 is the coefficient of $|v|^\alpha_V$ appearing in the growth of the operator B. However, the corresponding values required in main theorem of [3] would be less than 2/5 for Examples 2.1, 2.2 and less than $\frac{8(\alpha-1)}{\alpha^2} \wedge \frac{2(\alpha-1)}{5\alpha-1}$ for Example 2.4. Thus, the restriction on γ appearing in the growth assumption of operator B is not optimal in [3]. Further, operators B having growth like in Example 2.3 cannot be covered by [3].

Note that the restriction on the range of values γ may take, is not surprising in view of known results for linear stochastic partial differential equations where the "stochastic parabolicity" condition is needed. To see how this arises, consider the initial value problem

$$dv_t = (1 - \frac{1}{2}\gamma^2)\Delta v_t \, dt \ \text{ on } \ (0,T) \times \mathbb{R}^d ,$$

with $v_0 \in L^2(\mathbb{R}^d)$ given as an initial value. This is well-posed if $(1 - \frac{1}{2}\gamma^2) > 0$. Let $u_t(x) := v(t, x + \gamma W_t)$, where W is $\mathbb{R}$-valued Wiener process. Itô's formula implies that

$$du_t = \Delta u_t dt + \sum_{i=1}^d \gamma D_i u_t dW_t, \ \text{ on } \ (0,T) \times \mathbb{R}^d, \ \ u_0 = v_0.$$

Hence we can only reasonably expect this stochastic partial differential equation to be well-posed if $(1 - \frac{1}{2}\gamma^2) > 0$.

On the other hand, we see that the range of values of γ we may take, so that Assumption A-2.3 is satisfied, depends on p_0. This may seem surprising in view of results in Krylov [26] on L^p-theory for stochastic partial differential equations. The following example, which is not covered in [26], from Brzeźniak and Veraar [4], explores this question further.

Example 2.5. Consider the stochastic partial differential equation

$$du_t = \Delta u_t dt + 2\gamma(-\Delta)^{\frac{1}{2}} u_t \, dW_t \quad \text{on} \quad (0,T) \times \mathbb{T}, \tag{2.58}$$

where $\mathbb{T}$ is the one-dimensional torus $\mathbb{R}/(2\pi\mathbb{Z})$, $\gamma \in \mathbb{R}$ is a constant and $\mathcal{F}_0$-measurable u_0 is a given initial condition. Here W is a real-valued Wiener process.

For $\gamma^2 \in (0, 1/2)$ and $u_0 \in L^2(\Omega; L^2(\mathbb{T}))$, the results in Krylov and Rozovskii [28] imply existence and uniqueness of the solution to (2.58) and moreover the solution satisfies,

$$\mathbb{E} \sup_{t\in[0,T]} |u_t|^2_{L^2(\mathbb{T})} < C\mathbb{E}\Big(1 + |u_0|^2_{L^2(\mathbb{T})}\Big).$$

On the other hand Brzeźniak and Veraar [4] have shown that if

$$2\gamma^2(p-1) > 1,$$

then the problem (2.58) is not well-posed in $L^p((0,T)\times\Omega;L^2(\mathbb{T}))$. We now show that this example fits in the framework considered in this chapter and that the p_0-stochastic coercivity condition, Assumption A-2.3, is satisfied as long as

$$2\gamma^2(p_0-1) < 1. \tag{2.59}$$

This shows that the p_0-stochastic coercivity condition in this thesis is sharp, since (2.58) is ill-posed as soon as Assumption A-2.3 does not hold.

Let the space $L^2(\mathbb{T})$ denote the Lebesgue space of equivalence classes of $\mathbb{C}$-valued measurable functions u defined on any interval of length 2π, which are 2π-periodic and the norm

$$|u|_{L^2(\mathbb{T})} := \Big(\int_{\mathbb{T}} |u(x)|^2 dx\Big)^{\frac{1}{2}} < \infty\,.$$

Further, $W^{1,2}(\mathbb{T})$ denotes the closure of $C^\infty(\mathbb{T})$, the space of smooth functions, in $L^2(\mathbb{T})$ with respect to the norm $|\cdot|_{1,2}$ given by

$$|u|_{1,2} := \Bigg(\int_{\mathbb{T}} \big(|u(x)|^2 + |Du(x)|^2\big)dx\Bigg)^{\frac{1}{2}}.$$

Let $\mathcal{F}: L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ be the Fourier transform given by

$$\mathcal{F}u := (\hat{u}_k)_{k\in\mathbb{Z}} \text{ with } \hat{u}_k = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} u(x)e^{-ikx}dx$$

and $\mathcal{F}^{-1}: \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be its inverse, which is given by

$$\mathcal{F}^{-1}(\hat{u}_k)_{k\in\mathbb{Z}} =: u \text{ with } u(x) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}.$$

For $u \in W^{1,2}(\mathbb{T})$, we have

$$|u|^2_{W^{1,2}(\mathbb{T})} = |\mathcal{F}u|^2_{\ell^2(\mathbb{Z})} + |\mathcal{F}(Du)|^2_{\ell^2(\mathbb{Z})}, \quad \text{since} \quad |u|^2_{L^2(\mathbb{T})} = |\mathcal{F}u|^2_{\ell^2(\mathbb{Z})}. \tag{2.60}$$

Furthermore, for each $k \in \mathbb{Z}$,

$$[\mathcal{F}(Du)](k) = ik(\mathcal{F}u)(k). \tag{2.61}$$

Consider the operator $(-\Delta)^{\frac{1}{2}} : W^{1,2}(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$(-\Delta)^{\frac{1}{2}}u := \mathcal{F}^{-1}\Big(\big(|k|(\mathcal{F}u)(k)\big)_{k\in\mathbb{Z}}\Big)$$

and the operators $A: W^{1,2}(\mathbb{T}) \to W^{-1,2}(\mathbb{T})$ and $B: W^{1,2}(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$A(u) = \Delta u \text{ and } B(u) = 2\gamma(-\Delta)^{\frac{1}{2}}u.$$

It will be shown that these satisfy Assumptions A-2.1 to A-2.4. Using the arguments given in Example 2.1, the operator A satisfies Assumptions A-2.1 and A-2.4 with $\alpha = 2$, $\beta = 0$, $p_0 = 2$ and $L = 0$. Then, using (2.60) and (2.61), we obtain

$$2\langle A(u) - A(v), u - v\rangle + |B(u) - B(v)|^2_{L^2(\mathbb{T})} = (-2 + 4\gamma^2)\sum_{k\in\mathbb{Z}} k^2\big|(\mathcal{F}u)(k) - (\mathcal{F}v)(k)\big|^2 \le 0$$

provided $2\gamma^2 \le 1$. Hence operators A and B satisfy Assumption A-2.2 if $2\gamma^2 \le 1$. Furthermore, for any $\theta > 0$ and $p_0 \geq 2$, we obtain

$$2\langle A(u), u\rangle + (p_0 - 1)|Bu|^2_{L^2(\mathbb{T})} + \theta|u|^2_{W^{1,2}(\mathbb{T})} = (4\gamma^2(p_0 - 1) - 2 + \theta)\sum_{k\in\mathbb{Z}} k^2|(\mathcal{F}u)(k)|^2 + \theta|u|^2_{L^2(\mathbb{T})}.$$

Note that there is $\theta > 0$ such that $(4\gamma^2(p_0 - 1) - 2 + \theta) \leq 0$ if and only if $2\gamma^2(p_0 - 1) < 1$. Hence Assumption A-2.3 holds if and only if (2.59) holds and in this case, from Theorems 2.1, 2.2 and 2.3, we have that SPDE (2.58) has a unique solution. Moreover, the solution satisfies

$$\mathbb{E} \sup_{t\in[0,T]} |u_t|^p_{L^2(\mathbb{T})} < C\mathbb{E}\Big(1 + |u_0|^{p_0}_{L^2(\mathbb{T})}\Big)$$

for $p \in [2, p_0)$ if $p_0 > 2$ and for $p = 2$ otherwise.

Chapter 3

Lévy driven SPDEs with locally monotone coefficients

SPDEs driven by jump type noises have gained immense popularity in recent years as jump type noises can capture large unpredictable moves much better than the Wiener noise. Thus in this chapter, we extend the existence and uniqueness results of the previous chapter when the stochastic evolution equations under local monotonicity conditions are driven by Lévy noise. Further, the drift term is allowed to be the sum of finitely many operators each having different analytic and growth properties. As an application, we have shown the well-posedness of stochastic anisotropic p-Laplace equation driven by Lévy noise. Such an equation in deterministic setting has been considered by Lions [29]. This chapter is based on the results obtained in my article [38].

The chapter is organised as follows. In Section 3.1, we discuss the motivation behind considering the drift operator to be a finite sum of operators having different analytic and growth properties. In Section 3.2, we formulate and prove the main results of this chapter, see Theorems 3.1, 3.2 and 3.4. In Section 3.3, we show the well-posedness of the anisotropic p-Laplace equation (3.1) by proving Theorem 3.5. In Section 3.4, we give an example of a stochastic partial differential equation which fit into the framework of this chapter but, to the best of our knowledge, can not be solved by using results available so far. Finally in Section 3.5, we explain the interlacing procedure which allows one to construct the unique solution to an SPDE with large jumps from the unique solution of the corresponding SPDE with only small jumps.

3.1 Motivation

Consider the following stochastic anisotropic p-Laplace equation driven by Lévy noise,

$$\begin{aligned} du_t = &\sum_{i=1}^{d} D_i\big(|D_i u_t|^{p_i-2} D_i u_t\big)\, dt + \sum_{j=1}^{d} \zeta_j |D_j u_t|^{\frac{p_j}{2}}\, dW_t^j + \sum_{j=1}^{\infty} h_j(u_t) dW_t^j \\ &+ \int_{\mathcal{D}^c} \gamma_t(u_t,z)\tilde{N}(dt,dz) + \int_{\mathcal{D}} \gamma_t(u_t,z) N(dt,dz) \ \text{ on } \ (0,T)\times\mathscr{D}, \end{aligned} \tag{3.1}$$

where $u_t = 0$ on boundary of domain $\mathscr{D} \subset \mathbb{R}^d$ and u_0 is a given initial condition. Here, $p_i \geq 2$ are real numbers, ζ_j are constants and W^j are independent Wiener processes on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. Note that the Poisson random measure $N(dt,dz)$ defined on a σ-finite measure space $(Z, \mathscr{Z}, \nu)$, introduced in Chapter 1, is independent of the Wiener processes W^j. Further, $\mathcal{D} \in \mathscr{Z}$ is such that $\nu(\mathcal{D}) < \infty$ and $\mathcal{D}^c = Z \setminus \mathcal{D}$. The term anisotropic signifies that the parameter p in the p-Laplace operator takes different values in different directions, which is evident from the drift term of (3.1) as p_i's can be different. The precise assumptions on the functions h_j and γ are given in Theorem 3.5.

Solvability of anisotropic p-Laplace equation in deterministic setting, i.e.

$$du_t = \sum_{i=1}^{d} D_i\big(|D_i u_t|^{p_i-2} D_i u_t\big)\, dt \ \text{ on } \ (0,T)\times \mathscr{D}, \ \ u_t = 0 \text{ on } \partial\mathscr{D} \tag{3.2}$$

has been studied in Lions [29]. Note that if $p_i = p$ for all i, then a solution to (3.2) can be found in the Banach space defined by

$$W_0^{1,p}(\mathscr{D}) := \{u | u, D_i u \in L^p(\mathscr{D}),\ i = 1,2,\ldots,d;\ u = 0 \text{ on } \partial\mathscr{D}\}.$$

By solution we mean a function $u \in L^p((0,T); W_0^{1,p}(\mathscr{D}))$ such that for every $t \in [0,T]$ and $\phi \in W_0^{1,p}(\mathscr{D})$,

$$\int_{\mathscr{D}} u_t(x)\phi(x)dx = \int_{\mathscr{D}} u_0(x)\phi(x)dx - \sum_{i=1}^{d} \int_0^t \int_{\mathscr{D}} |D_i u_s(x)|^{p-2} D_i u_s(x) D_i\phi(x) dx ds\,.$$

The proof of existence of a solution to PDE (3.2), with $p_i = p$ for all i, uses the coercivity of the operator $\sum_{i=1}^{d} D_i\big(|D_i u|^{p-2} D_i u\big)$, which means there exists a constant $\theta > 0$, known as coefficient of coercivity, such that

$$-\sum_{i=1}^{d} \int_{\mathscr{D}} |D_i u(x)|^p dx \leq -\theta |u|^p_{W_0^{1,p}}\,.$$

However, when p_i's are different, we can not mimic the above argument as we can not find a p and a space X such that

$$-\sum_{i=1}^{d} \int_{\mathscr{D}} |D_i u(x)|^{p_i} dx \leq -\theta |u|^p_X$$

holds. To overcome this problem, Lions [29] considered the anisotropic p-Laplace operator $\sum_{i=1}^{d} D_i\big(|D_i u|^{p_i-2} D_i u\big)$ as a sum of d operators $D_i\big(|D_i u|^{p_i-2} D_i u\big)$, $i = 1,2,\ldots,d$, where each operator satisfies the coercivity condition with different p_i, θ_i and the space X_i, let's call it anisotropic coercivity condition. Then from the appropriate energy equality and anisotropic coercivity condition, the author gets the required a priori estimates. The usual compactness and monotonicity arguments lead to existence of a unique solution of (3.2) in the space $\cap_{i=1}^{d} L^{p_i}((0,T); W_0^{1,p_i}(\mathscr{D}))$. Pardoux [39] generalized the method of monotone operators used by Lions, and developed a theory for stochastic PDEs with monotone coercive operators. This theory can be applied to solve anisotropic p-Laplace equation driven by Wiener process. In this chapter, the technique used in [29] is extended so that anisotropic p-Laplace equation (3.1) driven by Lévy noise can be solved in a suitable space.

3.2 Assumptions and main results

Let $(H, (\cdot,\cdot), |\cdot|_H)$ be a separable Hilbert space, identified with its dual. For $i = 1,2,\ldots,k$, let $(V_i, |\cdot|_{V_i})$ be Banach spaces with duals $(V_i^*, |\cdot|_{V_i^*})$ and $\langle\cdot,\cdot\rangle_i$ be the notation for duality pairing between V_i and V_i^*. It is well known that the vector space $V := V_1 \cap V_2 \cap \ldots \cap V_k$ with the norm $|\cdot|_V := |\cdot|_{V_1} + |\cdot|_{V_2} \cdots + |\cdot|_{V_k}$ is a Banach space. Assume that V is separable, reflexive and is embedded continuously and densely in H. Thus we obtain the Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

where $\hookrightarrow$ denotes continuous and dense embedding.

We consider the stochastic evolution equation driven by Lévy noise of the following form:

$$du_t = \sum_{i=1}^{k} A_t^i(u_t)dt + \sum_{j=1}^{\infty} B_t^j(u_t)dW_t^j + \int_{\mathcal{D}^c} \gamma_t(u_t,z)\tilde{N}(dt,dz) + \int_{\mathcal{D}} \gamma_t(u_t,z)N(dt,dz) \tag{3.3}$$

for $t \in [0,T]$, where $\mathcal{D} \in \mathscr{Z}$ is such that $\nu(\mathcal{D}) < \infty$. Here, $A^i, i = 1,2,\dots,k$ are non-linear operators mapping $[0,T] \times \Omega \times V_i$ into V_i^*, $B = (B^j)_{j\in\mathbb{N}}$ is a non-linear operator mapping $[0,T]\times\Omega\times V$ into $\ell^2(H)$ and γ is a non-linear operator mapping $[0,T]\times\Omega\times V\times Z$ into H. Assume that for all $v,w \in V_i$, the processes $\big(\langle A_t^i(v),w\rangle_i\big)_{t\in[0,T]}$ are progressively measurable and for all $v,w\in V$, $\big((w,B_t^j(v))\big)_{t\in[0,T]}$ are progressively measurable. As mentioned earlier in Chapter 2, using Pettis' theorem, we obtain that for all $v\in V_i$, $i=1,2,\dots,k$, $\big(A_t^i(v)\big)_{t\in[0,T]}$ are progressively measurable. Further, for all $v\in V$, $j\in\mathbb{N}$, $\big(B_t^j(v)\big)_{t\in[0,T]}$ are progressively measurable. Finally, γ is assumed to be $\mathscr{P}\times\mathscr{B}(V)\times\mathscr{Z}$-measurable function and u_0 is assumed to be a given H-valued, $\mathscr{F}_0$-measurable random variable.

Further, we assume that there exist constants $\alpha_i > 1\,(i=1,2,\dots,k)$, $\beta\geq 0$, $p_0 \geq \beta+2$, $\theta > 0$, K, L', L'' and a nonnegative $f \in L^{\frac{p_0}{2}}((0,T)\times\Omega;\mathbb{R})$ such that, almost surely, the following conditions hold for all $t\in[0,T]$.

A - 3.1 (Hemicontinuity)**.** For $i=1,2,\dots,k$ and $y,x,\bar{x}\in V_i$, the map

$$\varepsilon \mapsto \langle A_t^i(x+\varepsilon\bar{x}),y\rangle_i$$

is continuous.

A - 3.2 (Local Monotonicity)**.** For all $x,\bar{x}\in V$,

$$\begin{aligned}2\sum_{i=1}^k\langle A_t^i(x)-A_t^i(\bar{x}),x-\bar{x}\rangle_i+\sum_{j=1}^\infty|B_t^j(x)-B_t^j(\bar{x})|_H^2+\int_{\mathcal{D}^c}|\gamma_t(x,z)-\gamma_t(\bar{x},z)|_H^2\nu(dz)\\ \leq\Big[L'+L''\Big(1+\sum_{i=1}^k|\bar{x}|_{V_i}^{\alpha_i}\Big)(1+|\bar{x}|_H^\beta)\Big]|x-\bar{x}|_H^2\,.\end{aligned}$$

A - 3.3 (p_0-Stochastic Coercivity)**.** For all $x\in V$,

$$2\sum_{i=1}^k\langle A_t^i(x),x\rangle_i+(p_0-1)\sum_{j=1}^\infty|B_t^j(x)|_H^2+\theta\sum_{i=1}^k|x|_{V_i}^{\alpha_i}+\int_{\mathcal{D}^c}|\gamma_t(x,z)|_H^2\nu(dz)\leq f_t+K|x|_H^2.$$

A - 3.4 (Growth of A^i)**.** For $i=1,2,\dots,k$ and $x\in V_i$,

$$|A_t^i(x)|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}}\leq(f_t+K|x|_{V_i}^{\alpha_i})(1+|x|_H^\beta).$$

A - 3.5 (Integrability of γ)**.** For all $x\in V$,

$$\int_{\mathcal{D}^c}|\gamma_t(x,z)|_H^{p_0}\nu(dz)\leq f_t^{\frac{p_0}{2}}+K|x|_H^{p_0}.$$

Remark 3.1. From Assumptions A-3.3 and A-3.4, we obtain

$$\sum_{j=1}^\infty|B_t^j(x)|_H^2+\int_{\mathcal{D}^c}|\gamma_t(x,z)|_H^2\nu(dz)\leq C\Big(1+f_t^{\frac{p_0}{2}}+|x|_H^{p_0}+\sum_{i=1}^k|x|_{V_i}^{\alpha_i}+|x|_H^\beta\sum_{i=1}^k|x|_{V_i}^{\alpha_i}\Big)$$

almost surely for all $t\in[0,T]$ and $x\in V$. Proof is very similar to the proof of Remark 2.1. Indeed, using Hölder's inequality, Young's inequality and Assumption A-3.4, we obtain that almost surely for all $x\in V$ and $t\in[0,T]$,

$$\begin{aligned}\sum_{i=1}^k|\langle A_t^i(x),x\rangle_i|&\leq\sum_{i=1}^k\Big[\frac{\alpha_i-1}{\alpha_i}|A_t^i(x)|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}}+\frac{1}{\alpha_i}|x|_{V_i}^{\alpha_i}\Big]\\&\leq\sum_{i=1}^k\Big[\frac{\alpha_i-1}{\alpha_i}\big(f_t+K|x|_{V_i}^{\alpha_i}\big)(1+|x|_H^\beta)+\frac{1}{\alpha_i}|x|_{V_i}^{\alpha_i}\Big]\end{aligned}$$

$$\leq C\Big(f_t + \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + |x|_H^\beta \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + f_t^{\frac{p_0}{2}} + (1+|x|_H)^{p_0}\Big).$$

The above inequality along with Assumption A-3.3 gives,

$$\begin{aligned}
&(p_0-1)\sum_{j=1}^\infty |B_t^j(x)|_H^2 + \int_{\mathcal{D}^c} |\gamma_t(x,z)|_H^2 \nu(dz) \\
&\quad \leq f_t + K|x|_H^2 - \theta \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + C\Big(f_t + \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + |x|_H^\beta \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + f_t^{\frac{p_0}{2}} + (1+|x|_H)^{p_0}\Big) \\
&\quad \leq C\Big((1+f_t)^{\frac{p_0}{2}} + \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + |x|_H^\beta \sum_{i=1}^k |x|_{V_i}^{\alpha_i} + f_t^{\frac{p_0}{2}} + (1+|x|_H)^{p_0}\Big)
\end{aligned}$$

and hence the result.

Further, in case $p_0 = 2$, i.e. $\beta = 0$, using the similar argument as above, we get

$$\sum_{j=1}^\infty |B_t^j(x)|_H^2 + \int_{\mathcal{D}^c} |\gamma_t(x,z)|_H^2 \nu(dz) \leq C\Big(f_t + |x|_H^2 + \sum_{i=1}^k |x|_{V_i}^{\alpha_i}\Big)$$

almost surely for all $t \in [0,T]$ and $x \in V$.

Remark 3.2. From Assumptions A-3.1, A-3.2 and A-3.4, we obtain that almost surely for all $t \in [0,T]$ and $i = 1,2,\dots,k$, the operators A_t^i are demicontinuous, i.e. $v_n \to v$ in V_i implies that $A_t^i(v_n) \rightharpoonup A_t^i(v)$ in V_i^*. This follows using similar arguments as in the proof of Lemma 2.1 in [28] or Remark 2.2 in this thesis. As a consequence, progressive measurability of some process $(v_t)_{t\in[0,T]}$ implies the progressive measurability of the processes $\big(A_t^i(v_t)\big)_{t\in[0,T]}$ for all $i = 1,2,\dots,k$.

If the driving noise in (3.3) is a Wiener process, i.e. intensity $\nu \equiv 0$, then Pardoux [39] has studied such equations when the operators satisfy hemicontinuity condition A-3.1, monotonicity condition (A-3.2 with constant $L'' = 0$), coercivity condition (A-3.3 with $p_0 = 2$, i.e. $\beta = 0$), growth assumption (A-3.4 with $\beta = 0$) and an additional assumption on operator B appearing in the stochastic integral term. Note that the noise considered in [39] is a cylindrical Q-Wiener process taking values in a separable Hilbert space. One can see, e.g. in Appendix A, that the stochastic Itô integral with respect to cylindrical Q-Wiener process taking values in a separable Hilbert space can be expressed in the form of infinite sum of stochastic Itô integrals with respect to independent one-dimensional Wiener processes as considered in (3.3). In view of this fact, the additional condition on operator B assumed in [39] can be equivalently stated as the following. For all $h \in H$ and positive real numbers N, there exists a constant M such that for almost all $(t,\omega) \in [0,T]\times\Omega$ and $x,y \in V$ satisfying $|x|_V, |y|_V \leq N$, it holds that

$$\sum_{j=1}^\infty |(h, B_t^j(x)) - (h, B_t^j(y))| \leq M|x-y|_V\,. \tag{3.4}$$

For the case $k = 1$, Krylov and Rozovskii [28] generalized the results in [39] by removing the additional assumption (3.4) on the operator B. As mentioned earlier in Chapter 1, the classical results in [28] have been generalised in number of directions. Gyöngy [11] extended the results in [28] to include SPDEs driven by càdlàg semi-martingales and thus allows ν in (2.1) to be different from zero. As discussed in Chapter 2, Liu and Röckner [30] have extended the framework in [28] to SPDEs with locally monotone operators where the operator A, which is the operator acting in the bounded variation term, satisfies a less restrictive growth condition. Thus, authors in [30] allow constants L'' and β, appearing in Assumptions A-3.2 and A-3.4 respectively, to be non-zero. Brzeźniak, Liu and Zhu [3] generalised the results in [30] to include equations driven by Lévy noise (i.e. $\nu \not\equiv 0$). However, authors in both [3] and [30] have placed an assumption on the growth of the operators appearing under stochastic integrals. Indeed, in the set up of this chapter, assumption made in [30] can be equivalently stated as: for all $(t,\omega) \in [0,T]\times\Omega$

and $x \in V$,

$$\sum_{j=1}^{\infty} |B_t^j(x)|_H^2 \leq C(f_t + |x|_H^2) \tag{3.5}$$

for some $f \in L^{\frac{p_0}{2}}((0,T) \times \Omega; \mathbb{R})$. Further, assumption made in [3] can be stated as: for $f \in L^{\frac{p_0}{2}}((0,T) \times \Omega; \mathbb{R})$, there exists a constant $\xi < \frac{\theta'}{2\beta}$ such that for all $(t, \omega) \in [0,T] \times \Omega$ and $x \in V$,

$$\sum_{j=1}^{\infty} |B_t^j(x)|_H^2 + \int_{\mathcal{D}^c} |\gamma_t(x,z)|_H^2 \nu(dz) \leq f_t + C|x|_H^2 + \xi |x|_V^\alpha \tag{3.6}$$

where θ' is the coefficient of coercivity appearing in coercivity assumption made in [3]. In view of Remark 3.1, the conditions (3.5) and (3.6) clearly place a restriction on the growth of operators appearing in stochastic integrals. As presented in Chapter 2, in our joint research work Neelima and Šiška [36], we have overcome this problem for the case $\nu \equiv 0$ by identifying the appropriate coercivity assumption as stated in Assumption A-3.3 and proved the existence and uniqueness of solutions to (3.3) (in case $k = 1$ and $\nu \equiv 0$) without explicitly restricting the growth of the operator B given in (3.5). The work presented in this chapter is a generalization of [3] in two senses: (a) we do not require the explicit growth condition (3.6) to establish existence and uniqueness results, (b) the operator acting in the bounded variation term is of the form $A^1 + A^2 + \cdots + A^k$, where the operators A^i have different analytic and growth properties. Again, we have generalized the results in [36] by including SPDEs driven by Lévy noise which satisfy condition (b) stated above, i.e. allowing $k > 1$ and $\nu \not\equiv 0$.

In all the above mentioned works, the key to prove the results is the use of an appropriate Itô's formula for the square of the H-norm. Here, we use the Itô's formula for processes taking values in intersection of finitely many Banach spaces, given recently by Gyöngy and Šiška [16] and extend the available results in the literature to include the SPDEs of the type (3.3) under the above mentioned assumptions.

Definition 3.1 (Solution)**.** An adapted, càdlàg, H-valued process u is called a solution of the stochastic evolution equation (3.3) if

i) $dt \times \mathbb{P}$ almost everywhere $u \in V$ with

$$\mathbb{E} \int_0^T (|u_t|_{V_i}^{\alpha_i} + |u_t|_H^2)\, dt < \infty\,, \qquad i = 1, 2, \ldots, k,$$

ii) for every $t \in [0,T]$ and $\phi \in V$,

$$\begin{aligned}(u_t, \phi) =& (u_0, \phi) + \sum_{i=1}^{k} \int_0^t \langle A_s(u_s), \phi \rangle_i ds + \sum_{j=1}^{\infty} \int_0^t (\phi, B_s^j(u_s)) dW_s^j \\ &+ \int_0^t \int_{\mathcal{D}^c} (\phi, \gamma_s(u_s, z)) \tilde{N}(ds, dz) + \int_0^t \int_{\mathcal{D}} (\phi, \gamma_s(u_s, z)) N(ds, dz)\end{aligned}$$

almost surely.

Note that the fact that u is càdlàg, H-valued process and i) in Definition 3.1 implies that almost surely,

$$\int_0^T \left(|u_t|_H^{p_0} + |u_t|_{V_i}^{\alpha_i} |u_t|_H^{p_0 - 2} \right) dt < \infty, \qquad i = 1, 2, \ldots, k.$$

The existence and uniqueness of solution to (3.3) can be obtained from the existence of a unique solution to the stochastic evolution equation,

$$u_t = u_0 + \sum_{i=1}^{k} \int_0^t A_s^i(u_s) ds + \sum_{j=1}^{\infty} \int_0^t B_s^j(u_s) dW_s^j + \int_0^t \int_{\mathcal{D}^c} \gamma_s(u_s, z) \tilde{N}(ds, dz) \tag{3.7}$$

for $t \in [0,T]$, i.e. the case when the last integral in (3.3) vanishes. This is done by means of the interlacing procedure (see e.g. [3, Section 4.2]). For the sake of completeness of argument, the procedure has been explained at the end of this chapter, see Section 3.5. As a consequence, we will now consider the stochastic evolution equation (3.7) in rest of the chapter and prove the existence and uniqueness of solution to (3.7) in Theorems 3.1, 3.2 and 3.4 below. We now show the existence and uniqueness of solution to SPDE (3.7).

3.2.1 A priori estimates

We begin by obtaining some a priori estimates of the solution to SPDE (3.7).

Theorem 3.1. *If u is a solution of* (3.7) *and Assumptions A-3.3 to A-3.5 hold, then for any $p_0 > 2$*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}|u_t|_H^{p_0} + \sum_{i=1}^k \mathbb{E}\int_0^T |u_t|_H^{p_0-2}|u_t|_{V_i}^{\alpha_i}dt \leq C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\,ds\Big) \\
&\textit{and} \qquad \sup_{t\in[0,T]} \mathbb{E}|u_t|_H^2 + \sum_{i=1}^k \mathbb{E}\int_0^T |u_t|_{V_i}^{\alpha_i}dt \leq C\mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s\,ds\Big)\,.
\end{aligned}
\tag{3.8}
$$

Moreover,

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|u_t|_H^2 \leq C\mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s ds\Big) \\
&\textit{and} \qquad \mathbb{E}\sup_{t\in[0,T]}|u_t|_H^{p_0 r} \leq C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\,ds\Big)^r
\end{aligned}
\tag{3.9}
$$

for any $r\in(0,1)$, where C depends only on p_0, K, T, r and θ.

Proof. Let u be a solution of (3.7). In order to obtain higher moment a priori estimates for solutions to (3.7), we define for each $n \in \mathbb{N}$,

$$
\sigma_n := \inf\{t\in[0,T] : |u_t|_H > n\} \wedge T. \tag{3.10}
$$

The solution u, being an adapted and càdlàg H-valued process, is bounded on every compact interval. Thus $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of stopping times converging to T, $\mathbb{P}$-a.s. and $\mathbb{P}\{\sigma_n < T\} = 0$ as $n\to\infty$. Applying Itô's formula for processes taking values in intersection of finitely many Banach spaces to (3.7), see [16, Theorem 2.1] and replacing t by $t\wedge\sigma_n$, we get almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
|u_{t\wedge\sigma_n}|_H^2 =& |u_0|_H^2 + \int_0^{t\wedge\sigma_n}\Big(2\sum_{i=1}^k\langle A_s^i(u_s),u_s\rangle_i + \sum_{j=1}^\infty |B_s^j(u_s)|_H^2\Big)ds \\
&+ 2\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}(u_s,B_s^j(u_s))dW_s^j + \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}2(u_s,\gamma_s(u_s,z))\tilde{N}(ds,dz) \\
&+ \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}|\gamma_s(u_s,z)|_H^2 N(ds,dz)\,.
\end{aligned}
\tag{3.11}
$$

Using the fact $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$, we get

$$
\begin{aligned}
|u_{t\wedge\sigma_n}|_H^2 =& |u_0|_H^2 + \int_0^{t\wedge\sigma_n}\Big(2\sum_{i=1}^k\langle A_s^i(u_s),u_s\rangle_i + \sum_{j=1}^\infty |B_s^j(u_s)|_H^2 + \int_{\mathcal{D}^c}|\gamma_s(u_s,z)|_H^2\nu(dz)\Big)ds \\
&+ 2\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}(u_s,B_s^j(u_s))dW_s^j + \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big(2(u_s,\gamma_s(u_s,z)) + |\gamma_s(u_s,z)|_H^2\Big)\tilde{N}(ds,dz)
\end{aligned}
$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$. Notice that this is a real-valued Itô process. Thus, by Itô's formula,

$$
\begin{aligned}
|u_{t\wedge\sigma_n}|_H^{p_0} =& |u_0|_H^{p_0} + \frac{p_0}{2}\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-2}\Big(2\sum_{i=1}^k \langle A_s^i(u_s),u_s\rangle_i + \sum_{j=1}^\infty |B_s^j(u_s)|_H^2 \\
&+ \int_{\mathcal{D}^c} |\gamma_s(u_s,z)|_H^2 \nu(dz)\Big)\,ds + p_0\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-2}\sum_{j=1}^\infty (u_s,B_s^j(u_s))dW_s^j \\
&+ \frac{p_0}{2}\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c} |u_s|_H^{p_0-2}\Big[2(u_s,\gamma_s(u_s,z)) + |\gamma_s(u_s,z)|_H^2\Big]\tilde N(ds,dz) \\
&+ \frac{p_0(p_0-2)}{2}\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-4}\sum_{j=1}^\infty |(u_s,B_s^j(u_s))|^2\,ds \\
&+ \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[\big||u_s|_H^2 + 2(u_s,\gamma_s(u_s,z)) + |\gamma_s(u_s,z)|_H^2\big|^{\frac{p_0}{2}} - |u_s|_H^{p_0} \\
&- \frac{p_0}{2}|u_s|_H^{p_0-2}\big[2(u_s,\gamma_s(u_s,z)) + |\gamma_s(u_s,z)|_H^2\big]\Big]N(ds,dz)
\end{aligned}
$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$. Again, using the fact $\tilde N(dt,dz) = N(dt,dz) - \nu(dz)dt$, we get

$$
\begin{aligned}
|u_{t\wedge\sigma_n}|_H^{p_0} = |u_0|_H^{p_0} + I_1 + I_2 + p_0\sum_{j=1}^\infty \int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-2}(u_s,B_s^j(u_s))dW_s^j \\
+ p_0\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c} |u_s|_H^{p_0-2}(u_s,\gamma_s(u_s,z))\tilde N(ds,dz)
\end{aligned}
\tag{3.12}
$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$, where

$$
\begin{aligned}
I_1 :=& \frac{p_0}{2}\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-2}\Big(2\sum_{i=1}^k \langle A_s^i(u_s),u_s\rangle_i + \sum_{j=1}^\infty |B_s^j(u_s)|_H^2\Big)\,ds \\
&+ \frac{p_0(p_0-2)}{2}\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-4}\sum_{j=1}^\infty |(u_s,B_s^j(u_s))|^2\,ds
\end{aligned}
$$

and

$$
I_2 := \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s+\gamma_s(u_s,z)|_H^{p_0} - |u_s|_H^{p_0} - p_0|u_s|_H^{p_0-2}(u_s,\gamma_s(u_s,z))\Big]N(ds,dz)\,.
$$

Using Cauchy-Schwarz inequality, Assumption A-3.3 and Young's inequality, we get almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
I_1 &\le \frac{p_0}{2}\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-2}\Big(2\sum_{i=1}^k \langle A_s^i(u_s),u_s\rangle_i + (p_0-1)\sum_{j=1}^\infty |B_s^j(u_s)|_H^2\Big)ds \\
&\le \frac{p_0}{2}\int_0^{t\wedge\sigma_n} |u_s|_H^{p_0-2}\Big(f_s + K|u_s|_H^2 - \theta\sum_{i=1}^k |u_s|_{V_i}^{\alpha_i}\Big)ds \\
&\le \int_0^{t\wedge\sigma_n}\Big(f_s^{\frac{p_0}{2}} + \frac{p_0(K+1)-2}{2}|u_s|_H^{p_0} - \theta\frac{p_0}{2}\sum_{i=1}^k |u_s|_H^{p_0-2}|u_s|_{V_i}^{\alpha_i}\Big)ds\,.
\end{aligned}
\tag{3.13}
$$

We now proceed to estimate I_2. Notice that due to Taylor's formula on the map $t\mapsto |x+ty|_H^p$, for any $x,y\in H$ and $p\ge 2$, we get

$$
|x+y|_H^p - |x|_H^p = \int_0^1 \frac{d}{dt}|x+ty|_H^p dt
$$

and therefore,

$$
\begin{aligned}
\big||x+y|_H^p - |x|_H^p - p|x|_H^{p-2}(x,y)\big| &= p\Big|\int_0^1 \big[|x+ty|_H^{p-2}(x+ty,y) - |x|_H^{p-2}(x,y)\big]\,dt\Big| \\
&\le C_p\int_0^1 \big(|x|_H^{p-2}+|y|_H^{p-2}\big)|y|_H^2\,t\,dt \le C_p(|x|_H^{p-2}|y|_H^2 + |y|_H^p)\,.
\end{aligned}
\tag{3.14}
$$

Now, taking $x = u_s$, $y = \gamma_s(u_s,z)$ and $p = p_0$ in (3.14), we get

$$
|u_s+\gamma_s(u_s,z)|_H^{p_0} - |u_s|_H^{p_0} - p_0|u_s|_H^{p_0-2}(u_s,\gamma_s(u_s,z)) \le C\Big(|u_s|_H^{p_0-2}|\gamma_s(u_s,z)|_H^2 + |\gamma_s(u_s,z)|_H^{p_0}\Big)
$$

and hence using Young's inequality, we get for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
I_2 \le& C\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s|_H^{p_0-2}|\gamma_s(u_s,z)|_H^2 + |\gamma_s(u_s,z)|_H^{p_0}\Big]N(ds,dz) \\
\le& C\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s|_H^{p_0} + |\gamma_s(u_s,z)|_H^{p_0}\Big]N(ds,dz)\,.
\end{aligned}
\tag{3.15}
$$

Using (3.13) and (3.15), we obtain from (3.12)

$$
\begin{aligned}
|u_{t\wedge\sigma_n}|_H^{p_0} + \theta\frac{p_0}{2}\sum_{i=1}^k\int_0^{t\wedge\sigma_n}|u_s|_H^{p_0-2}|u_s|_{V_i}^{\alpha_i}ds \le |u_0|_H^{p_0} + \int_0^{t\wedge\sigma_n} f_s^{\frac{p_0}{2}}\,ds + C\int_0^{t\wedge\sigma_n}|u_s|_H^{p_0}ds \\
+ C\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\big[|u_s|_H^{p_0} + |\gamma_s(u_s,z)|_H^{p_0}\big]N(ds,dz) \\
+ p_0\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}|u_s|_H^{p_0-2}(u_s,B_s^j(u_s))dW_s^j \\
+ p_0\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}|u_s|_H^{p_0-2}(u_s,\gamma_s(u_s,z))\tilde N(ds,dz)
\end{aligned}
\tag{3.16}
$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$. We now aim to apply Lemma 1.9. To that end let τ be some bounded stopping time. Then in view of Remark 3.1 and the fact that u is a solution of equation (3.7), it follows that for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$
\mathbb{E}\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}\mathbf{1}_{\{s\le\tau\}}|u_s|_H^{p_0-2}(u_s,B_s^j(u_s))dW_s^j = 0
$$

and

$$
\mathbb{E}\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\mathbf{1}_{\{s\le\tau\}}|u_s|_H^{p_0-2}(u_s,\gamma_s(u_s,z))\tilde N(ds,dz) = 0\,.
$$

Therefore, replacing $t\wedge\sigma_n$ by $t\wedge\sigma_n\wedge\tau$ in (3.16), taking expectation and using Assumption A-3.5, we obtain for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} + \theta\frac{p_0}{2}\sum_{i=1}^k\mathbb{E}\int_0^{t\wedge\sigma_n\wedge\tau}|u_s|_H^{p_0-2}|u_s|_{V_i}^{\alpha_i}ds \le \mathbb{E}|u_0|_H^{p_0} + \mathbb{E}\int_0^T f_s^{\frac{p_0}{2}}\,ds \\
+ \mathbb{E}\int_0^{t\wedge\sigma_n\wedge\tau}|u_s|_H^{p_0}ds + C\mathbb{E}\int_0^{t\wedge\sigma_n\wedge\tau}\int_{\mathcal{D}^c}\big[|u_s|_H^{p_0} + |\gamma_s(u_s,z)|_H^{p_0}\big]\nu(dz)ds \\
\le \mathbb{E}|u_0|_H^{p_0} + C\mathbb{E}\int_0^T f_s^{\frac{p_0}{2}}\,ds + C\mathbb{E}\int_0^t |u_{s\wedge\sigma_n\wedge\tau}|_H^{p_0}ds
\end{aligned}
\tag{3.17}
$$

From this Gronwall's lemma yields,

$$
\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau}|_H^{p_0} \le C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\,ds\Big)
\tag{3.18}
$$

for all $t \in [0,T]$ and $n \in \mathbb{N}$. Letting $n \to \infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}|u_{t\wedge\tau}|_H^{p_0} \le C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\, ds\Big)$$

for all $t \in [0,T]$. Using Lemma 1.9, we get

$$\mathbb{E}\sup_{t\in[0,T]}|u_t|_H^{p_0 r} \le \frac{2-r}{1-r}C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\, ds\Big)^r$$

for any $r \in (0,1)$, which proves the second inequality in (3.9).

In order to prove (3.8), the estimate (3.18) is used in the right-hand side of (3.17) with $\tau = T$ and with $n \to \infty$. We thus obtain,

$$\mathbb{E}|u_t|_H^{p_0} + \theta\frac{p_0}{2}\sum_{i=1}^k \mathbb{E}\int_0^t |u_s|_H^{p_0-2}|u_s|_{V_i}^{\alpha_i} ds \le C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\, ds\Big)$$

for all $t \in [0,T]$. If Assumption A-3.3 holds for some $p_0 \ge \beta + 2$, then it holds for $p_0 = 2$ as well. Thus, from (3.11) we obtain

$$\mathbb{E}|u_t|_H^2 + \theta\sum_{i=1}^k \mathbb{E}\int_0^t |u_s|_{V_i}^{\alpha_i} ds \le \mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s ds\Big) + K\mathbb{E}\int_0^t |u_s|_H^2 ds$$

for all $t \in [0,T]$. Application of Gronwall's lemma yields,

$$\sup_{t\in[0,T]}\mathbb{E}|u_t|_H^2 \le C\mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s ds\Big),$$

which in turn gives

$$\theta\sum_{i=1}^k \mathbb{E}\int_0^T |u_s|_{V_i}^{\alpha_i} ds \le C\mathbb{E}\Big(|u_0|_H^2 + \int_0^T f_s\, ds\Big)$$

and hence (3.8) holds.

To complete the proof it remains to show the first inequality in (3.9). Considering the sequence of stopping times σ_n defined in (3.10), as before we observe that the stochastic integrals appearing in the right-hand side of (3.11) are martingales for each $n \in \mathbb{N}$. Thus using the Burkholder–Davis–Gundy inequality and Cauchy–Schwarz inequality, we obtain for each $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^\infty \int_0^{t\wedge\sigma_n} (u_s, B_s^j(u_s))dW_s^j\Big| \\ &\le 4\mathbb{E}\Big(\sum_{j=1}^\infty\int_0^{T\wedge\sigma_n} |(u_s, B_s^j(u_s))|^2 ds\Big)^{\frac{1}{2}} \\ &\le 4\mathbb{E}\Big(\sum_{j=1}^\infty\int_0^{T\wedge\sigma_n} |u_s|_H^2|B_s^j(u_s)|_H^2 ds\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.19}$$

Similarly, using Lemma 1.5 and Cauchy-Schwarz inequality, we obtain for each $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c} (u_s, \gamma_s(u_s))\tilde{N}(ds,dz)\Big| \\ &\le C\mathbb{E}\Big(\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c} |(u_s, \gamma_s(u_s))|^2\nu(dz)ds\Big)^{\frac{1}{2}} \\ &\le C\mathbb{E}\Big(\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c} |u_s|_H^2|\gamma_s(u_s)|_H^2\nu(dz)ds\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.20}$$

Thus (3.19), (3.20) along with Remark 3.1 and Young's inequality give,

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}(u_s,B_s^j(u_s))dW_s^j\Big| &+ \mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}(u_s,\gamma_s(u_s))\tilde{N}(ds,dz)\Big| \\
&\leq C\mathbb{E}\Big(\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\int_0^{T\wedge\sigma_n}\big(f_s+|u_s|_H^2+\sum_{i=1}^k|u_s|_{V_i}^{\alpha_i}\big)ds\Big)^{\frac{1}{2}} \qquad (3.21)\\
&\leq \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2+C\mathbb{E}\int_0^{T\wedge\sigma_n}\big(f_s+|u_s|_H^2+\sum_{i=1}^k|u_s|_{V_i}^{\alpha_i}\big)ds\,,
\end{aligned}
$$

for each $n\in\mathbb{N}$. Moreover, taking supremum and then expectation in (3.11) and using Assumption A-3.3 along with (3.21), we obtain for each $n\in\mathbb{N}$

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 \leq& \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 \\
&+C\Big(\mathbb{E}|u_0|_H^2+\mathbb{E}\int_0^T f_s\,ds+\sum_{i=1}^k\mathbb{E}\int_0^T|u_s|_{V_i}^{\alpha_i}ds+\sup_{t\in[0,T]}\mathbb{E}|u_t|_H^2\Big).
\end{aligned}
$$

Finally, by choosing ϵ small and using (3.8) for $p_0=2$, we obtain for each $n\in\mathbb{N}$

$$
\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\leq C\Big(\mathbb{E}|u_0|_H^2+\mathbb{E}\int_0^T f_s\,ds\Big)
$$

which on allowing $n\to\infty$ and using Fatou's lemma finishes the proof. □

Note that we can obtain existence and uniqueness results even if Assumption A-3.3 is replaced by the following assumption.

A - 3.6. For all $x\in V$,

$$
2\sum_{i=1}^k\langle A_t^i(x),x\rangle_i+(p_0-1)\sum_{j=1}^{\infty}|B_t^j(x)|_H^2+\theta\sum_{i=1}^k[x]_{V_i}^{\alpha_i}+\int_{\mathcal{D}^c}|\gamma_t(x,z)|_H^2\nu(dz)\leq f_t+K|x|_H^2\,,
$$

where, $\alpha_i<p_0$ for all i and $[\cdot]_{V_i}$ is a seminorm on the space V_i such that

$$
|\cdot|_{V_i}\leq|\cdot|_H+[\cdot]_{V_i}\,.
$$

Indeed, in next remark we show that we obtain apriori estimates similar to (3.8) even if Assumption A-3.3 is replaced by A-3.6 and then rest of the argument for showing existence and uniqueness of solution to (3.7) will remain the same.

Remark 3.3. If Assumption A-3.3 is replaced by the A-3.6, then replacing $|u_t|_{V_i}^{\alpha_i}$ by $[u_t]_{V_i}^{\alpha_i}$ everywhere in the proof of Theorem 3.1, we obtain

$$
\sum_{i=1}^k\mathbb{E}\int_0^T[u_s]_{V_i}^{\alpha_i}ds\leq C\mathbb{E}\Big(|u_0|_H^2+\int_0^T f_s\,ds\Big).
$$

Also,

$$
\mathbb{E}\int_0^T|u_s|_H^{\alpha_i}ds\leq T\mathbb{E}\sup_{s\in[0,T]}|u_s|_H^{\alpha_i}\leq C\mathbb{E}\Big(|u_0|_H^{p_0}+\int_0^T f_s^{\frac{p_0}{2}}\,ds\Big)
$$

since $\alpha_i<p_0$ for all i.

Thus,

$$
\sum_{i=1}^k\mathbb{E}\int_0^T|u_s|_{V_i}^{\alpha_i}ds\leq\sum_{i=1}^k C\Big(\mathbb{E}\int_0^T|u_s|_H^{\alpha_i}ds+\mathbb{E}\int_0^T[u_s]_{V_i}^{\alpha_i}ds\Big)
$$

$$\leq C\mathbb{E}\Big(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}}\,ds + |u_0|_H^2 + \int_0^T f_s ds\Big)$$

giving all the desired a priori estimates for the solution.

3.2.2 Uniqueness of solution

Before stating the result about uniqueness of solution to stochastic evolution equation (3.7), we observe the following.

We note that right hand side in the Assumption A-3.2 can be replaced by

$$\Big[L\Big(1 + \sum_{i=1}^{k} |\bar{x}|_{V_i}^{\alpha_i}\Big)(1 + |\bar{x}|_H^{\beta})\Big]|x - \bar{x}|_H^2$$

for some constant L. We use this L in the remaining chapter.

Definition 3.2. Let Ψ be defined as the collection of V-valued and $\mathscr{F}_t$-adapted processes ψ satisfying

$$\int_0^T \rho(\psi_s)\,ds\, < \infty \qquad \text{a.s.}\,,$$

where,

$$\rho(x) := L\Big(1 + \sum_{i=0}^{k} |x|_{V_i}^{\alpha_i}\Big)(1 + |x|_H^{\beta})$$

for all $x \in V$.

Note that if u is a solution to (3.7) then $u \in \Psi$.

Remark 3.4. For any $\psi \in \Psi$ and $v \in L^2(\Omega, D([0,T]; H))$,

$$\begin{aligned}\mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)dr}\rho(\psi_s)|v_s|_H^2 ds\Big] &\leq \mathbb{E}\sup_{s\in[0,t]}|v_s|_H^2 \int_0^t e^{-\int_0^s \rho(\psi_r)dr}\rho(\psi_s)ds \\ &= \mathbb{E}\sup_{s\in[0,t]}|v_s|_H^2[1 - e^{-\int_0^t \rho(\psi_r)dr}] \leq \mathbb{E}\sup_{s\in[0,t]}|v_s|_H^2 < \infty.\end{aligned}$$

This remark justifies the existence of the bounded variation integrals appearing in the proof of uniqueness that follows.

Theorem 3.2. *Let Assumptions A-3.2 to A-3.5 hold and $u_0, \bar{u}_0 \in L^{p_0}(\Omega; H)$. If u and $\bar{u}$ are two solutions of (3.7) with $u_0 = \bar{u}_0$ $\mathbb{P}$-a.s., then the processes u and $\bar{u}$ are indistinguishable, i.e.*

$$\mathbb{P}\Big(\sup_{t\in[0,T]}|u_t - \bar{u}_t|_H = 0\Big) = 1.$$

Proof. Consider two solutions u and $\bar{u}$ of (3.7). Thus,

$$\begin{aligned}u_t - \bar{u}_t =& u_0 - \bar{u}_0 + \sum_{i=1}^{k}\int_0^t \big(A_s^i(u_s) - A_s^i(\bar{u}_s)\big)\,ds + \sum_{j=1}^{\infty}\int_0^t \big(B_s^j(u_s) - B_s^j(\bar{u}_s)\big)\,dW_s^j \\ &+ \int_0^t \int_{\mathcal{D}^c} (\gamma_s(u_s, z) - \gamma_s(\bar{u}_s, z))\tilde{N}(ds, dz)\end{aligned} \tag{3.22}$$

almost surely for all $t \in [0, T]$. Using the product rule and the Itô's formula from [16], we obtain

$$\begin{aligned}d\Big(e^{-\int_0^t \rho(\bar{u}_s)\,ds}|u_t - \bar{u}_t|_H^2\Big) &= e^{-\int_0^t \rho(\bar{u}_s)ds}\big[d|u_t - \bar{u}_t|_H^2 - \rho(\bar{u}_t)|u_t - \bar{u}_t|_H^2\,dt\big] \\ =& e^{-\int_0^t \rho(\bar{u}_s)ds}\Big[\Big(2\sum_{i=1}^{k}\langle A_t^i(u_t) - A_t^i(\bar{u}_t), u_t - \bar{u}_t\rangle_i + \sum_{j=1}^{\infty}|B_t^j(u_t) - B_t^j(\bar{u}_t)|_H^2\Big)\,dt\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{\infty} 2\big(u_t-\bar u_t, B_t^j(u_t)-B_t^j(\bar u_t)\big)dW_t^j+\int_{\mathcal{D}^c} 2(u_t-\bar u_t,\gamma_t(u_t,z)-\gamma_t(\bar u_t,z))\tilde N(dt,dz)\\
&+\int_{\mathcal{D}^c}|\gamma_t(u_t,z)-\gamma_t(\bar u_t,z)|_H^2N(dt,dz)-\rho(\bar u_t)|u_t-\bar u_t|_H^2dt\Big]
\end{aligned}
\tag{3.23}
$$

almost surely for all $t\in[0,T]$. For each $n\in\mathbb{N}$, consider the sequence of stopping times σ_n given by

$$
\sigma_n:=\inf\{t\in[0,T]:|u_t|_H>n\}\wedge\inf\{t\in[0,T]:|\bar u_t|_H>n\}\wedge T\,. \tag{3.24}
$$

Replacing t by $t_n:=t\wedge\sigma_n$ in (3.23) and taking expectation, we obtain that almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
&\mathbb{E}\Big(e^{-\int_0^{t_n}\rho(\bar u_s)\,ds}|u_{t_n}-\bar u_{t_n}|_H^2\Big)-\mathbb{E}|u_0-\bar u_0|_H^2\\
&\quad=\mathbb{E}\int_0^{t_n}e^{-\int_0^s\rho(\bar u_r)dr}\Big(2\sum_{i=1}^k\langle A_s^i(u_s)-A_s^i(\bar u_s),u_s-\bar u_s\rangle_i+\sum_{j=1}^{\infty}|B_s^j(u_s)-B_s^j(\bar u_s)|_H^2\\
&\qquad+\int_{\mathcal{D}^c}|\gamma_s(u_s,z)-\gamma_s(\bar u_s,z)|_H^2\nu(dz)-\rho(\bar u_s)|u_s-\bar u_s|_H^2\Big)\,ds\le 0
\end{aligned}
$$

where last inequality follows from Assumption A-3.2. Thus if $u_0=\bar u_0$ $\mathbb{P}$-a.s., then

$$
\mathbb{E}[e^{-\int_0^{t_n}\rho(\bar u_s)ds}|u_{t_n}-\bar u_{t_n}|_H^2]\le 0.
$$

Letting $n\to\infty$ and using Fatou's lemma we conclude that $\mathbb{P}(|u_t-\bar u_t|_H^2=0)=1$ for all $t\in[0,T]$. This, together with the fact that $u-\bar u$ is càdlàg in H, finishes the proof. □

As in Chapter 2, in order to get results about continuous dependence of the solution to (3.7) on the initial data, we consider the following.

A - 3.7 (Strong Monotonicity)**.** There exist constants $\theta'>0$ and K such that almost surely, for all $t\in[0,T]$ and $x,\bar x\in V$,

$$
\begin{aligned}
&2\sum_{i=1}^k\langle A_t^i(x)-A_t^i(\bar x),x-\bar x\rangle_i+(p_0-1)\sum_{j=1}^{\infty}|B_t^j(x)-B_t^j(\bar x)|_H^2\\
&+\int_{\mathcal{D}^c}|\gamma_t(x,z)-\gamma_t(\bar x,z)|_H^2\nu(dz)\le-\theta'\sum_{i=1}^k|x-\bar x|_{V_i}^{\alpha_i}+K|x-\bar x|_H^2\,.
\end{aligned}
$$

A - 3.8. There exist a constant K such that almost surely, for all $t\in[0,T]$ and $x,\bar x\in V$,

$$
\int_{\mathcal{D}^c}|\gamma_t(x,z)-\gamma_t(\bar x,z)|_H^{p_0}\nu(dz)\le K|x-\bar x|_H^{p_0}.
$$

A - 3.9. There exist a constant K such that almost surely, for all $t\in[0,T]$ and $x,\bar x\in V$,

$$
\sum_{j=1}^{\infty}|B_t^j(x)-B_t^j(\bar x)|_H^2+\int_{\mathcal{D}^c}|\gamma_t(x,z)-\gamma_t(\bar x,z)|_H^2\nu(dz)\le K\Big(|x-\bar x|_H^2+\sum_{i=1}^k|x-\bar x|_{V_i}^{\alpha_i}\Big).
$$

If we replace the local monotonicity Assumption A-3.2 by the strong monotonicity Assumption A-3.7 and Assumption A-3.5 by Assumption A-3.8, then we obtain the following result about the continuous dependence of the solution to (3.7) on the initial data.

Theorem 3.3. *Let Assumptions A-3.3, A-3.4, A-3.7 and A-3.8 hold and $u_0,\bar u_0\in L^{p_0}(\Omega;H)$. If u and $\bar u$ are two solutions of* (3.7) *with initial condition u_0 and $\bar u_0$ respectively, then for $p_0>2$*

$$
\sup_{t\in[0,T]}\mathbb{E}|u_t-\bar u_t|_H^{p_0}+\mathbb{E}\sum_{i=1}^k\int_0^T|u_t-\bar u_t|_H^{p_0-2}|u_t-\bar u_t|_{V_i}^{\alpha_i}dt<C\mathbb{E}|u_0-\bar u_0|_H^{p_0}\,,
$$

$$\mathbb{E}\sup_{t\in[0,T]}|u_t-\bar{u}_t|_H^{p_0 r}<C\mathbb{E}|u_0-\bar{u}_0|_H^{p_0 r}$$

for any $r\in(0,1)$ *and*

$$\sup_{t\in[0,T]}\mathbb{E}|u_t-\bar{u}_t|_H^2+\sum_{i=1}^k\mathbb{E}\int_0^T|u_t-\bar{u}_t|_{V_i}^{\alpha_i}dt<C\mathbb{E}|u_0-\bar{u}_0|_H^2\,.$$

Proof. As in Chapter 2, the result is obtained by applying Itô's formula from [16] to (3.22) and repeating the proof of Theorem 3.1 for the process $u_t-\bar{u}_t$. Here we note that one needs to use the strong monotonicity Assumption A-3.7 in place of Assumption A-3.3, Assumption A-3.8 in place of Assumption A-3.5 and work with the sequence of stopping times given by (3.24). We include the proof for the convenience of reader.

Let u and $\bar{u}$ be two solutions of (3.7) and thus (3.22) holds. For each $n\in\mathbb{N}$, consider the sequence of stopping times σ_n given by (3.24). The process $u-\bar{u}$, being an adapted and càdlàg H-valued process, is bounded on every compact interval. Thus $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of stopping times converging to T, $\mathbb{P}$- a.s. and $\mathbb{P}\{\sigma_n<T\}=0$ as $n\to\infty$. Applying Itô's formula from [16] to (3.22) and replacing t by $t\wedge\sigma_n$, we get almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$\begin{aligned}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2&=|u_0-\bar{u}_0|_H^2+\int_0^{t\wedge\sigma_n}\Big(2\sum_{i=1}^k\langle A_s^i(u_s)-A_s^i(\bar{u}_s),u_s-\bar{u}_s\rangle_i\\&+\sum_{j=1}^\infty|B_s^j(u_s)-B_s^j(\bar{u}_s)|_H^2\Big)ds+2\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}(u_s-\bar{u}_s,B_s^j(u_s)-B_s^j(\bar{u}_s))dW_s^j\\&+\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}2(u_s-\bar{u}_s,\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z))\tilde{N}(ds,dz)\\&+\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2N(ds,dz),\end{aligned}\tag{3.25}$$

which on using the fact $\tilde{N}(dt,dz)=N(dt,dz)-\nu(dz)dt$, gives

$$\begin{aligned}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2&=|u_0-\bar{u}_0|_H^2+\int_0^{t\wedge\sigma_n}\Big(2\sum_{i=1}^k\langle A_s^i(u_s)-A_s^i(\bar{u}_s),u_s-\bar{u}_s\rangle_i\\&+\sum_{j=1}^\infty|B_s^j(u_s)-B_s^j(\bar{u}_s)|_H^2+\int_{\mathcal{D}^c}|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2\nu(dz)\Big)ds\\&+2\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}(u_s-\bar{u}_s,B_s^j(u_s)-B_s^j(\bar{u}_s))dW_s^j\\&+\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big(2(u_s-\bar{u}_s,\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z))+|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2\Big)\tilde{N}(ds,dz)\end{aligned}$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$. Notice that this is a real-valued Itô process. Thus, by Itô's formula,

$$\begin{aligned}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^{p_0}&=|u_0-\bar{u}_0|_H^{p_0}+\frac{p_0}{2}\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}\Big(2\sum_{i=1}^k\langle A_s^i(u_s)-A_s^i(\bar{u}_s),u_s-\bar{u}_s\rangle_i\\&+\sum_{j=1}^\infty|B_s^j(u_s)-B_s^j(\bar{u}_s)|_H^2+\int_{\mathcal{D}^c}|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2\nu(dz)\Big)\,ds\\&+p_0\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}\sum_{j=1}^\infty(u_s-\bar{u}_s,B_s^j(u_s)-B_s^j(\bar{u}_s))dW_s^j\\&+\frac{p_0}{2}\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}|u_s-\bar{u}_s|_H^{p_0-2}\Big[2(u_s-\bar{u}_s,\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z))+|\gamma_s(u_s,z)\end{aligned}$$

$$- \gamma_s(\bar{u}_s,z)|_H^2\Big]\tilde{N}(ds,dz) + \frac{p_0(p_0-2)}{2}\int_0^{t\wedge\sigma_n} |u_s - \bar{u}_s|_H^{p_0-4}\sum_{j=1}^{\infty}|(u_s-\bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s))|^2\,ds$$

$$+ \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[\big||u_s-\bar{u}_s|_H^2 + 2(u_s-\bar{u}_s, \gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)) + |\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2\big|^{\frac{p_0}{2}}$$
$$- |u_s-\bar{u}_s|_H^{p_0} - \frac{p_0}{2}|u_s-\bar{u}_s|_H^{p_0-2}\big[2\big(u_s-\bar{u}_s, \gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)\big)$$
$$+ |\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2\big]\Big]N(ds,dz)$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$. Again, using the fact $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$, we get

$$|u_{t\wedge\sigma_n} - \bar{u}_{t\wedge\sigma_n}|_H^{p_0} = |u_0-\bar{u}_0|_H^{p_0} + p_0\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}(u_s-\bar{u}_s, B_s^j(u_s)-B_s^j(\bar{u}_s))dW_s^j$$
$$+ I_1 + I_2 + p_0\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}|u_s-\bar{u}_s|_H^{p_0-2}\big(u_s-\bar{u}_s, \gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)\big)\tilde{N}(ds,dz) \tag{3.26}$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$, where

$$I_1 := \frac{p_0}{2}\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}\Big(2\sum_{i=1}^{k}\langle A_s^i(u_s)-A_s^i(\bar{u}_s), u_s-\bar{u}_s\rangle_i + \sum_{j=1}^{\infty}|B_s^j(u_s)-B_s^j(\bar{u}_s)|_H^2\Big)\,ds$$
$$+ \frac{p_0(p_0-2)}{2}\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-4}\sum_{j=1}^{\infty}|(u_s-\bar{u}_s, B_s^j(u_s)-B_s^j(\bar{u}_s))|^2\,ds$$

and

$$I_2 := \int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s-\bar{u}_s+\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0} - |u_s-\bar{u}_s|_H^{p_0}$$
$$- p_0|u_s-\bar{u}_s|_H^{p_0-2}\big(u_s-\bar{u}_s, \gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)\big)\Big]N(ds,dz)\,.$$

Using Cauchy-Schwarz inequality, Assumption A-3.7 and Young's inequality, we get almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$I_1 \le \frac{p_0}{2}\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}\Big(2\sum_{i=1}^{k}\langle A_s^i(u_s)-A_s^i(\bar{u}_s), u_s-\bar{u}_s\rangle_i$$
$$+ (p_0-1)\sum_{j=1}^{\infty}|B_s^j(u_s)-B_s^j(\bar{u}_s)|_H^2\Big)ds$$
$$\le \frac{p_0}{2}\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}\Big(K|u_s-\bar{u}_s|_H^2 - \theta'\sum_{i=1}^{k}|u_s-\bar{u}_s|_{V_i}^{\alpha_i}\Big)ds$$
$$\le \int_0^{t\wedge\sigma_n}\Big(K\frac{p_0}{2}|u_s-\bar{u}_s|_H^{p_0} - \theta'\frac{p_0}{2}\sum_{i=1}^{k}|u_s-\bar{u}_s|_H^{p_0-2}|u_s-\bar{u}_s|_{V_i}^{\alpha_i}\Big)ds\,. \tag{3.27}$$

In order to estimate I_2, we take $x = u_s-\bar{u}_s$, $y = \gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)$ and $p=p_0$ in (3.14). Thus, we get

$$|u_s-\bar{u}_s+\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0} - |u_s-\bar{u}_s|_H^{p_0} - p_0|u_s-\bar{u}_s|_H^{p_0-2}(u_s-\bar{u}_s, \gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z))$$
$$\le C\Big(|u_s-\bar{u}_s|_H^{p_0-2}|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2 + |\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0}\Big)$$

and hence using Young's inequality, we get for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$I_2 \le C\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s-\bar{u}_s|_H^{p_0-2}|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^2 + |\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0}\Big]N(ds,dz)$$

$$\leq C\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s-\bar{u}_s|_H^{p_0}+|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0}\Big]N(ds,dz)\,. \tag{3.28}$$

Using (3.27) and (3.28), we obtain from (3.26)

$$\begin{aligned}
|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^{p_0}&+\theta'\frac{p_0}{2}\sum_{i=1}^k\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}|u_s-\bar{u}_s|_{V_i}^{\alpha_i}ds\\
&\leq|u_0-\bar{u}_0|_H^{p_0}+\int_0^{t\wedge\sigma_n}K\frac{p_0}{2}|u_s-\bar{u}_s|_H^{p_0}ds\\
&\quad+C\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\Big[|u_s-\bar{u}_s|_H^{p_0}+|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0}\Big]N(ds,dz)\\
&\quad+p_0\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}|u_s-\bar{u}_s|_H^{p_0-2}(u_s-\bar{u}_s,B_s^j(u_s)-B_s^j(\bar{u}_s))dW_s^j\\
&\quad+p_0\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}|u_s-\bar{u}_s|_H^{p_0-2}(u_s-\bar{u}_s,\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z))\tilde{N}(ds,dz)
\end{aligned} \tag{3.29}$$

almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}$. We now aim to apply Lemma 1.9. To that end let τ be some bounded stopping time. Then in view of Remark 3.1 and the fact that u and $\bar{u}$ are solutions of equation (3.7), it follows that for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$\mathbb{E}\sum_{j=1}^\infty\int_0^{t\wedge\sigma_n}\mathbf{1}_{\{s\leq\tau\}}|u_s-\bar{u}_s|_H^{p_0-2}(u_s-\bar{u}_s,B_s^j(u_s)-B_s^j(\bar{u}_s))dW_s^j=0$$

and

$$\mathbb{E}\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}\mathbf{1}_{\{s\leq\tau\}}|u_s-\bar{u}_s|_H^{p_0-2}(u_s-\bar{u}_s,\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z))\tilde{N}(ds,dz)=0\,.$$

Therefore, replacing $t\wedge\sigma_n$ by $t\wedge\sigma_n\wedge\tau$ in (3.29), taking expectation and using Assumption A-3.8, we obtain for all $t\in[0,T]$ and $n\in\mathbb{N}$,

$$\begin{aligned}
\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau}-\bar{u}_{t\wedge\sigma_n\wedge\tau}|_H^{p_0}&+\theta'\frac{p_0}{2}\sum_{i=1}^k\mathbb{E}\int_0^{t\wedge\sigma_n\wedge\tau}|u_s-\bar{u}_s|_H^{p_0-2}|u_s-\bar{u}_s|_{V_i}^{\alpha_i}ds\\
&\leq\mathbb{E}|u_0-\bar{u}_0|_H^{p_0}+K\frac{p_0}{2}\mathbb{E}\int_0^{t\wedge\sigma_n\wedge T}|u_s-\bar{u}_s|_H^{p_0}ds\\
&\quad+C\mathbb{E}\int_0^{t\wedge\sigma_n\wedge\tau}\int_{\mathcal{D}^c}\Big[|u_s-\bar{u}_s|_H^{p_0}+|\gamma_s(u_s,z)-\gamma_s(\bar{u}_s,z)|_H^{p_0}\Big]\nu(dz)ds\\
&\leq\mathbb{E}|u_0-\bar{u}_0|_H^{p_0}+C\mathbb{E}\int_0^t|u_{s\wedge\sigma_n\wedge\tau}-\bar{u}_{s\wedge\sigma_n\wedge\tau}|_H^{p_0}ds\,.
\end{aligned} \tag{3.30}$$

From this Gronwall's lemma yields,

$$\mathbb{E}|u_{t\wedge\sigma_n\wedge\tau}-\bar{u}_{t\wedge\sigma_n\wedge\tau}|_H^{p_0}\leq C\mathbb{E}|u_0-\bar{u}_0|_H^{p_0} \tag{3.31}$$

for all $t\in[0,T]$ and $n\in\mathbb{N}$. Letting $n\to\infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}|u_{t\wedge\tau}-\bar{u}_{t\wedge\tau}|_H^{p_0}\leq C\mathbb{E}|u_0-\bar{u}_0|_H^{p_0}$$

for all $t\in[0,T]$. Using Lemma 1.9, we get

$$\mathbb{E}\sup_{t\in[0,T]}|u_t-\bar{u}_t|_H^{p_0r}\leq\frac{2-r}{1-r}C\mathbb{E}|u_0-\bar{u}_0|_H^{p_0r}$$

for any $r\in(0,1)$.

Further, using the estimate (3.31) in the right-hand side of (3.30) with $\tau=T$ and with

$n \to \infty$, we obtain

$$\mathbb{E}|u_t - \bar{u}_t|_H^{p_0} + \theta' \frac{p_0}{2} \sum_{i=1}^{k} \mathbb{E} \int_0^t |u_s - \bar{u}_s|_H^{p_0-2} |u_s - \bar{u}_s|_{V_i}^{\alpha_i} ds \leq C\mathbb{E}|u_0 - \bar{u}_0|_H^{p_0}$$

for all $t \in [0,T]$ as desired.

If Assumption A-3.7 holds for some $p_0 \geq \beta + 2$, then it holds for $p_0 = 2$ as well. Thus, from (3.25) with $n \to \infty$, we obtain

$$\mathbb{E}|u_t - \bar{u}_t|_H^2 + \theta' \sum_{i=1}^{k} \mathbb{E} \int_0^t |u_s - \bar{u}_s|_{V_i}^{\alpha_i} ds \leq \mathbb{E}\Big(|u_0 - \bar{u}_0|_H^2\Big) + K\mathbb{E}\int_0^t |u_s - \bar{u}_s|_H^2 ds$$

for all $t \in [0,T]$. Application of Gronwall's lemma yields

$$\sup_{t\in[0,T]} \mathbb{E}|u_t - \bar{u}_t|_H^2 \leq C\mathbb{E}|u_0 - \bar{u}_0|_H^2 \,, \tag{3.32}$$

which in turn gives

$$\theta' \sum_{i=1}^{k} \mathbb{E} \int_0^T |u_s - \bar{u}_s|_{V_i}^{\alpha_i} ds \leq C\mathbb{E}|u_0 - \bar{u}_0|_H^2 \tag{3.33}$$

and hence the result. □

Remark 3.5. Assuming Assumption A-3.9 in addition to assumptions made in Theorem 3.3 above, we further obtain

$$\mathbb{E} \sup_{t\in[0,T]} |u_t - \bar{u}_t|_H^2 < C\mathbb{E}|u_0 - \bar{u}_0|_H^2 \,.$$

Indeed, by considering the sequence of stopping times σ_n defined in (3.24), as earlier we observe that the stochastic integrals appearing in the right-hand side of (3.25) are martingales for each $n \in \mathbb{N}$. Thus using the Burkholder–Davis–Gundy inequality and Cauchy–Schwarz inequality, we obtain for each $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \Big| \sum_{j=1}^{\infty} \int_0^{t\wedge\sigma_n} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j \Big| \\
&\qquad \leq 4\mathbb{E}\Big(\sum_{j=1}^{\infty} \int_0^{T\wedge\sigma_n} |(u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s))|^2 ds\Big)^{\frac{1}{2}} \\
&\qquad \leq 4\mathbb{E}\Big(\sum_{j=1}^{\infty} \int_0^{T\wedge\sigma_n} |u_s - \bar{u}_s|_H^2 |B_s^j(u_s) - B_s^j(\bar{u}_s)|_H^2 ds\Big)^{\frac{1}{2}} .
\end{aligned} \tag{3.34}$$

Similarly, for each $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\sigma_n} \int_{\mathcal{D}^c} (u_s - \bar{u}_s, \gamma_s(u_s,z) - \gamma_s(\bar{u}_s,z)) \tilde{N}(ds,dz) \Big| \\
&\qquad \leq C\mathbb{E}\Big(\int_0^{T\wedge\sigma_n} \int_{\mathcal{D}^c} |(u_s - \bar{u}_s, \gamma_s(u_s,z) - \gamma_s(\bar{u}_s,z))|^2 \nu(dz) ds\Big)^{\frac{1}{2}} \\
&\qquad \leq C\mathbb{E}\Big(\int_0^{T\wedge\sigma_n} \int_{\mathcal{D}^c} |u_s - \bar{u}_s|_H^2 |\gamma_s(u_s,z) - \gamma_s(\bar{u}_s,z)|_H^2 \nu(dz) ds\Big)^{\frac{1}{2}} .
\end{aligned} \tag{3.35}$$

Thus (3.34), (3.35) along with Assumption A-3.9 and Young's inequality give,

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \Big| \sum_{j=1}^{\infty} \int_0^{t\wedge\sigma_n} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j \Big| \\
&\quad + \mathbb{E} \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\sigma_n} \int_{\mathcal{D}^c} (u_s - \bar{u}_s, \gamma_s(u_s,z) - \gamma_s(\bar{u}_s,z)) \tilde{N}(ds,dz) \Big|
\end{aligned}$$

$$
\begin{aligned}
&\le C\mathbb{E}\Big[\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2\int_0^{T\wedge\sigma_n}\Big(|u_s-\bar{u}_s|_H^2+\sum_{i=1}^k|u_s-\bar{u}_s|_{V_i}^{\alpha_i}\Big)ds\Big]^{\frac{1}{2}} \\
&\le\epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2+C\mathbb{E}\int_0^{T\wedge\sigma_n}\Big(|u_s-\bar{u}_s|_H^2+\sum_{i=1}^k|u_s-\bar{u}_s|_{V_i}^{\alpha_i}\Big)ds\,,
\end{aligned}
\tag{3.36}
$$

for each $n\in\mathbb{N}$. Moreover, taking supremum and then expectation in (3.25) and using Assumption A-3.7 along with (3.36), we obtain for each $n\in\mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2\le\epsilon\mathbb{E}&\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2 \\
&+C\Big(\mathbb{E}|u_0-\bar{u}_0|_H^2+\sum_{i=1}^k\mathbb{E}\int_0^T|u_s-\bar{u}_s|_{V_i}^{\alpha_i}ds+\sup_{t\in[0,T]}\mathbb{E}|u_t-\bar{u}_t|_H^2\Big).
\end{aligned}
$$

Finally, by choosing ϵ small and using (3.32) and (3.33), we obtain for each $n\in\mathbb{N}$,

$$
\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}-\bar{u}_{t\wedge\sigma_n}|_H^2\le C\mathbb{E}|u_0-\bar{u}_0|_H^2
$$

which on allowing $n\to\infty$ and using Fatou's lemma gives the desired result.

3.2.3 Existence of solution

We prove the existence of solution to SEE (3.7) by using the Galerkin method. We consider a Galerkin scheme $(\mathcal{V}_m)_{m\in\mathbb{N}}$ for V, i.e. for each $m\in\mathbb{N}$, $\mathcal{V}_m$ is an m-dimensional subspace of V such that $\mathcal{V}_m\subset\mathcal{V}_{m+1}\subset V$ and $\cup_{m\in\mathbb{N}}\mathcal{V}_m$ is dense in V. Let $\{\phi_l : l=1,2,\dots m\}$ be a basis of $\mathcal{V}_m$. Assume that for each $m\in\mathbb{N}$, u_0^m is a $\mathcal{V}_m$-valued $\mathscr{F}_0$-measurable random variable satisfying

$$
\sup_{m\in\mathbb{N}}\mathbb{E}|u_0^m|_H^{p_0}<\infty\ \text{ and }\ \mathbb{E}|u_0^m-u_0|_H^2\to0\ \text{ as }\ m\to\infty. \tag{3.37}
$$

As mentioned earlier in Chapter 2, it is possible to obtain such an approximating sequence.

For each $m\in\mathbb{N}$ and $\phi_l\in\mathcal{V}_m$, $l=1,2,\dots,m$, consider the stochastic differential equation:

$$
\begin{aligned}
(u_t^m,\phi_l)=&(u_0^m,\phi_l)+\sum_{i=1}^k\int_0^t\langle A_s^i(u_s^m),\phi_l\rangle_i ds \\
&+\sum_{j=1}^m\int_0^t(\phi_l,B_s^j(u_s^m))dW_s^j+\int_0^t\int_{\mathcal{D}^c}(\phi_l,\gamma_s(u_s^m,z))\tilde{N}(ds,dz)
\end{aligned}
\tag{3.38}
$$

almost surely for all $t\in[0,T]$. Using the results on solvability of stochastic differential equations in finite dimensional space (see, e.g., Theorem 1 in Gyöngy and Krylov [12]), together with Assumptions A-3.1 to A-3.5 and Remark 3.2, there exists a unique adapted and càdlàg (and thus progressively measurable) $\mathcal{V}_m$-valued process u^m satisfying (3.38).

Lemma 3.1 (A priori Estimates for Galerkin Discretization)**.** *Suppose that* (3.37) *and Assumptions A-3.3, A-3.4 and A-3.5 hold. Then there exists a constant C independent of m, such that*

i) for every $p_0\ge\beta+2$,

$$
\sup_{t\in[0,T]}\mathbb{E}|u_t^m|_H^{p_0}+\sum_{i=1}^k\mathbb{E}\int_0^T|u_t^m|_{V_i}^{\alpha_i}dt+\sum_{i=1}^k\mathbb{E}\int_0^T|u_t^m|_H^{p_0-2}|u_t^m|_{V_i}^{\alpha_i}dt\le C.
$$

ii) Further,

$$
\mathbb{E}\sup_{t\in[0,T]}|u_t^m|_H^2\le C\quad\textit{and}\quad\mathbb{E}\sup_{t\in[0,T]}|u_t^m|_H^p\le C
$$

for any $p\in[2,p_0)$, $p_0>2$.

iii) Moreover, for all $i = 1, 2, \dots, k$

$$\mathbb{E}\int_0^T |A_s^i(u_s^m)|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}} ds \le C$$

iv) and finally,

$$\mathbb{E}\sum_{j=1}^{\infty}\int_0^T |B_s^j(u_s^m)|_H^2 ds + \mathbb{E}\int_0^T\int_{\mathcal{D}^c} |\gamma_s(u_s^m, z)|_H^2 \nu(dz)ds \le C.$$

Proof. Proof of (i) and (ii) is almost a repetition of the proof of analogous results in Theorem 3.1. Indeed, for each $m, n \in \mathbb{N}$, one can define a sequence of stopping times

$$\sigma_n^m := \inf\{t \in [0,T] : |u_t^m|_H > n\} \wedge T$$

and repeat the proof of Theorem 3.1 by replacing u_t with u_t^m and σ_n with σ_n^m.

Here we recall two main points.

First, the stochastic integrals appearing on right-hand side of (3.11), with u_s replaced by u_s^m, are martingales for each $m, n \in \mathbb{N}$. Indeed, on a finite dimensional space, all norms are equivalent and hence for each $m, n \in \mathbb{N}$,

$$\mathbb{E}\int_0^{T\wedge\sigma_n^m} |u_s^m|_V^\alpha ds \le C_m \mathbb{E}\int_0^{T\wedge\sigma_n^m} n^\alpha ds < \infty$$

with some constant C_m.

The second point is that, since

$$\sup_{m\in\mathbb{N}} \mathbb{E}|u_0^m|^{p_0} < \infty,$$

one can take a constant independent of m to obtain (i) and (ii).

The estimates in (iii) and (iv) can be proved as below.

Using Assumption A-3.4, we obtain

$$\begin{aligned}
I &:= \sum_{i=1}^k \mathbb{E}\int_0^T |A_s^i(u_s^m)|_{V_i^*}^{\frac{\alpha_i}{\alpha_i-1}} ds \\
&\le \sum_{i=1}^k \mathbb{E}\int_0^T (f_s + K|u_s^m|_{V_i}^{\alpha_i})(1 + |u_s^m|_H^\beta) ds \\
&= k\mathbb{E}\int_0^T f_s\, ds + k\mathbb{E}\int_0^T f_s|u_s^m|_H^\beta ds + K\sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i} ds + K\sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_H^\beta |u_s^m|_{V_i}^{\alpha_i}\, ds\,.
\end{aligned}$$

Further application of Young's inequality yields,

$$\begin{aligned}
f_s + f_s|u_s^m|_H^\beta &\le \frac{4}{p_0} f_s^{\frac{p_0}{2}} + \frac{p_0-2}{p_0} + \frac{p_0-2}{p_0}|u_s^m|_H^{\beta\frac{p_0}{p_0-2}} \\
&\le \frac{4}{p_0} f_s^{\frac{p_0}{2}} + \frac{p_0-2}{p_0} + \frac{p_0-2-\beta}{p_0-2} + \frac{\beta}{p_0}|u_s^m|_H^{p_0}\,,
\end{aligned}$$

where we have used the fact that $p_0 \ge \beta + 2$. This also implies $|u_s^m|_H^\beta \le (1 + |u_s^m|_H)^{p_0-2}$. Hence,

$$\begin{aligned}
I \le C\Big[&\mathbb{E}\int_0^T f_s^{\frac{p_0}{2}} ds + T + \mathbb{E}\int_0^T |u_s^m|_H^{p_0} ds + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i} ds \\
&\sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}(1 + |u_s^m|_H)^{p_0-2} ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big[\mathbb{E}\int_0^T f_s^{\frac{p_0}{2}}\,ds + T + T\sup_{0\le s\le T}\mathbb{E}|u_s^m|_H^{p_0} \\
&\qquad + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}\,ds + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}|u_s^m|_H^{p_0-2}ds\Big].
\end{aligned} \tag{3.39}
$$

By using (i) in (3.39), we obtain (iii). Furthermore, by Remark 3.1, we get

$$
\begin{aligned}
\mathbb{E}&\int_0^T \sum_{j=1}^\infty |B_s^j(u_s^m)|_H^2 ds + \mathbb{E}\int_0^T\int_{\mathcal{D}^c} |\gamma_s(u_s^m,z)|_H^2\nu(dz)ds \\
&\le C\Big[T + \mathbb{E}\int_0^T f_s^{\frac{p_0}{2}}\,ds + \mathbb{E}\int_0^T |u_s^m|_H^{p_0}ds \\
&\qquad + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}ds + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}(1+|u_s^m|_H)^{p_0-2}ds\Big] \\
&\le C\Big[T + \mathbb{E}\int_0^T f_s^{\frac{p_0}{2}}\,ds + T\sup_{s\in[0,T]}\mathbb{E}|u_s^m|_H^{p_0} \\
&\qquad + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}ds + \sum_{i=1}^k \mathbb{E}\int_0^T |u_s^m|_{V_i}^{\alpha_i}|u_s^m|_H^{p_0-2}ds\Big]
\end{aligned}
$$

and hence by using (i), we get (iv). □

Having obtained the necessary a priori estimates, we will now extract weakly convergent subsequences using the compactness argument. After that using the local monotonicity condition, we establish the existence of a solution to (3.7).

Lemma 3.2. *Let Assumptions A-3.2 to A-3.5 together with* (3.37) *hold. Then there is a subsequence* $(m_q)_{q\in\mathbb{N}}$ *and*

i) there exists a process $u \in \cap_{i=1}^k L^{\alpha_i}((0,T)\times\Omega;V_i)$ *such that*

$$u^{m_q} \rightharpoonup u \ \text{ in } \ L^{\alpha_i}((0,T)\times\Omega;V_i) \quad \forall\, i=1,2,\ldots,k,$$

ii) there exist processes $a^i \in L^{\frac{\alpha_i}{\alpha_i-1}}((0,T)\times\Omega;V^*)$ *such that*

$$A^i(u^{m_q}) \rightharpoonup a^i \ \text{ in } \ L^{\frac{\alpha_i}{\alpha_i-1}}((0,T)\times\Omega;V^*) \quad \forall\, i=1,2,\ldots,k,$$

iii) there exists a process $b \in L^2((0,T)\times\Omega;\ell^2(H))$ *such that*

$$B(u^{m_q}) \rightharpoonup b \ \text{ in } \ L^2((0,T)\times\Omega;\ell^2(H)),$$

iv) there exists $\Gamma \in L^2((0,T)\times\Omega\times Z;H)$ *such that*

$$\gamma(u^{m_q})1_{\mathcal{D}^c} \rightharpoonup \Gamma 1_{\mathcal{D}^c} \ \text{ in } \ L^2((0,T)\times\Omega\times Z;H).$$

Proof. The Banach spaces $L^{\alpha_i}((0,T)\times\Omega;V_i)$, $L^{\frac{\alpha_i}{\alpha_i-1}}((0,T)\times\Omega;V_i^*)$, $L^2((0,T)\times\Omega;\ell^2(H))$ and $L^2((0,T)\times\Omega\times Z;H)$ are reflexive. Thus, due to Lemma 3.1, there exists a subsequence m_q (see, e.g., Theorem 3.18 in [2]) such that

(i) $u^{m_q} \rightharpoonup u^i$ in $L^{\alpha_i}((0,T)\times\Omega;V_i)$ $\forall\, i=1,2,\ldots,k$,

(ii) $A^i(u^{m_q}) \rightharpoonup a^i$ in $L^{\frac{\alpha_i}{\alpha_i-1}}((0,T)\times\Omega;V_i^*)$ $\forall\, i=1,2,\ldots,k$,

(iii) $(B^j(u^{m_q}))_{j=1}^{m_q} \rightharpoonup (b^j)_{j=1}^\infty$ in $L^2((0,T)\times\Omega;\ell^2(H))$,

(iv) $\gamma(u^{m_q})1_{\mathcal{D}^c} \rightharpoonup \Gamma 1_{\mathcal{D}^c}$ in $L^2((0,T)\times\Omega\times Z;H)$.

Further, for any $\xi \in V$ and for any adapted and bounded real valued process η_t, we have for $i, j \in \{1, 2, \dots, k\}$

$$\mathbb{E}\int_0^T \eta_t(u_t^i - u_t^j, \xi)dt = \mathbb{E}\int_0^T \eta_t(u_t^i - u_t^{m_q}, \xi)dt + \mathbb{E}\int_0^T \eta_t(u_t^{m_q} - u_t^j, \xi)dt$$

with right–hand–side converging to zero as $q \to \infty$. Therefore the processes u^i, $i = 1, 2, \dots, k$ are equal $dt \times \mathbb{P}$ almost everywhere and henceforth are denoted by u in the remaining chapter. □

Lemma 3.3. *Let Assumptions A-3.2 to A-3.5 together with* (3.37) *hold. Then*

i) for $dt \times \mathbb{P}$ almost everywhere,

$$u_t = u_0 + \sum_{i=1}^k \int_0^t a_s^i ds + \sum_{j=1}^\infty \int_0^t b_s^j dW_s^j + \int_0^t \int_{\mathcal{D}^c} \Gamma_s(z)\tilde{N}(ds, dz)$$

and moreover almost surely $u \in D([0, T]; H)$ and for all $t \in [0, T]$,

$$\begin{aligned}|u_t|_H^2 =& |u_0|_H^2 + \int_0^t \Big[2\sum_{i=1}^k \langle a_s^i, u_s\rangle + \sum_{j=1}^\infty |b_s^j|_H^2\Big] ds + 2\sum_{j=1}^\infty \int_0^t (u_s, b_s^j)dW_s^j \\ &+ \int_0^t \int_{\mathcal{D}^c} 2(u_s, \Gamma_s(z))\tilde{N}(ds, dz) + \int_0^t \int_{\mathcal{D}^c} |\Gamma_s(z)|_H^2 N(ds, dz)\,.\end{aligned} \tag{3.40}$$

ii) Finally, $u \in L^2\big(\Omega; D([0, T]; H)\big)$.

Proof. Using Itô's isometry, it can be shown that the stochastic integral with respect to Wiener process is a bounded linear operator from $L^2((0, T) \times \Omega; \ell^2(H))$ to $L^2((0, T) \times \Omega; H)$ and hence maps a weakly convergent sequence to a weakly convergent sequence. Thus, we obtain

$$\sum_{j=1}^{m_q} \int_0^\cdot B_s^j(u_s^{m_q})dW_s^j \rightharpoonup \sum_{j=1}^\infty \int_0^\cdot b_s^j dW_s^j$$

in $L^2([0, T] \times \Omega; H)$, i.e. for any $\psi \in L^2((0, T) \times \Omega; H)$,

$$\mathbb{E}\int_0^T \Big(\sum_{j=1}^{m_q} \int_0^t B_s^j(u_s^{m_q})dW_s^j, \psi(t)\Big)dt \to \mathbb{E}\int_0^T \Big(\sum_{j=1}^\infty \int_0^t b_s^j dW_s^j, \psi(t)\Big)dt. \tag{3.41}$$

By similar argument, for any $\psi \in L^2((0, T) \times \Omega; H)$ we have

$$\begin{aligned}&\mathbb{E}\int_0^T \Big(\int_0^t \int_{\mathcal{D}^c} \gamma_s(u_s^{m_q}, z)\tilde{N}(ds, dz), \psi(t)\Big)dt \\ &\qquad\to \mathbb{E}\int_0^T \Big(\int_0^t \int_{\mathcal{D}^c} \Gamma_s(z)\tilde{N}(ds, dz), \psi(t)\Big)dt.\end{aligned} \tag{3.42}$$

Similarly, using Holder's inequality it can be shown that for each $i = 1, 2, \dots, k$, the Bochner integral is a bounded linear operator from $L^{\frac{\alpha_i}{\alpha_i - 1}}((0, T) \times \Omega; V_i^*)$ to $L^{\frac{\alpha_i}{\alpha_i - 1}}((0, T) \times \Omega; V_i^*)$ and is thus continuous with respect to weak topologies. Therefore, for any $\psi \in L^{\alpha_i}((0, T) \times \Omega; V_i)$,

$$\mathbb{E}\int_0^T \Big\langle \int_0^t A_s^i(u_s^{m_q})ds, \psi(t)\Big\rangle dt \to \mathbb{E}\int_0^T \Big\langle \int_0^t a_s^i ds, \psi(t)\Big\rangle dt. \tag{3.43}$$

Fix $n \in \mathbb{N}$. Then for any $\phi \in \mathcal{V}_n$ and an adapted real valued process η_t bounded by a constant

C, we have for any $q \geq n$,

$$\mathbb{E}\int_0^T \eta_t(u_t^{m_q}, \phi)\,dt = \mathbb{E}\int_0^T \eta_t\Big[(u_0^{m_q}, \phi) + \sum_{i=1}^k \int_0^t \langle A_s^i(u_s^{m_q}), \phi\rangle\,ds$$
$$+ \sum_{j=1}^{m_q}\int_0^t (\phi, B_s^j(u_s^{m_q}))dW_s^j + \int_0^t\int_{\mathcal{D}^c}(\phi, \gamma_s(u_s^{m_q}, z))\tilde{N}(ds, dz)\Big]\,dt.$$

Taking the limit $q \to \infty$ and using (3.37), (3.41), (3.42) and (3.43), we obtain

$$\mathbb{E}\int_0^T \eta_t(u_t, \phi)\,dt = \mathbb{E}\int_0^T \eta_t\Big[(u_0, \phi) + \sum_{i=1}^k \int_0^t \langle a_s^i, \phi\rangle\,ds$$
$$+ \sum_{j=1}^{\infty}\int_0^t (\phi, b_s^j)\,dW_s^j + \int_0^t\int_{\mathcal{D}^c}(\phi, \Gamma_s(z))\tilde{N}(ds, dz)\Big]\,dt$$

with any $\phi \in \mathcal{V}_n$ and any adapted and bounded real valued process η_t. Since $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is dense in V, we obtain

$$u_t = u_0 + \sum_{i=1}^k \int_0^t a_s^i ds + \sum_{j=1}^{\infty}\int_0^t b_s^j dW_s^j + \int_0^t\int_{\mathcal{D}^c}\Gamma_s(z)\tilde{N}(ds, dz) \tag{3.44}$$

$dt \times \mathbb{P}$ almost everywhere.

Using Theorem 2.1 on Itô's formula from [16], there exists an H-valued càdlàg modification of the process u, denoted again by u, which is equal to the right hand side of (3.44) almost surely for all $t \in [0, T]$.

Moreover (3.40) holds almost surely for all $t \in [0, T]$. This completes the proof of part (i) of the lemma.

It remains to prove part (ii) of the lemma. To that end, consider the sequence of stopping times σ_n defined in (3.10). Using Burkholder–Davis–Gundy inequality together with Cauchy–Schwarz's and Young's inequalities, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^{\infty}\int_0^{t\wedge\sigma_n}(u_s, b_s^j)dW_s^j\Big| \leq 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|(u_s, b_s^j)|_H^2 ds\Big)^{\frac{1}{2}}\\
&\leq 4\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|u_s|_H^2|b_s^j|_H^2 ds\Big)^{\frac{1}{2}} \leq 4\mathbb{E}\Big(\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|b_s^j|_H^2 ds\Big)^{\frac{1}{2}}\\
&\leq \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 + C\mathbb{E}\sum_{j=1}^{\infty}\int_0^{T\wedge\sigma_n}|b_s^j|_H^2 ds.
\end{aligned} \tag{3.45}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n}\int_{\mathcal{D}^c}(u_s, \Gamma_s(z))\tilde{N}(ds, dz)\Big| &\leq C\mathbb{E}\Big(\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c}|(u_s, \Gamma_s(z))|_H^2\nu(dz)ds\Big)^{\frac{1}{2}}\\
&\leq C\mathbb{E}\Big(\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c}|u_s|_H^2|\Gamma_s(z)|_H^2\nu(dz)ds\Big)^{\frac{1}{2}}\\
&\leq C\mathbb{E}\Big(\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c}|\Gamma_s(z)|_H^2\nu(dz)ds\Big)^{\frac{1}{2}}\\
&\leq \epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 + C\mathbb{E}\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c}|\Gamma_s(z)|_H^2\nu(dz)ds.
\end{aligned} \tag{3.46}$$

Replace t by $t \wedge \sigma_n$ in (3.40) and take supremum and then expectation. On using Hölder's inequality along with (3.45) and (3.46), we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 \leq \mathbb{E}|u_0|_H^2 + 2\sum_{i=1}^k\Big(\mathbb{E}\int_0^T |a_s^i|^{\frac{\alpha_i}{\alpha_i-1}}ds\Big)^{\frac{\alpha_i-1}{\alpha_i}}\Big(\mathbb{E}\int_0^T |u_s|_{V_i}^{\alpha_i}ds\Big)^{\frac{1}{\alpha_i}}$$
$$+\epsilon\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 + C\mathbb{E}\sum_{j=1}^\infty\int_0^T|b_s^j|_H^2ds + C\mathbb{E}\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c}|\Gamma_s(z)|_H^2\nu(dz)ds$$

which on choosing ϵ small enough gives,

$$\mathbb{E}\sup_{t\in[0,T]}|u_{t\wedge\sigma_n}|_H^2 \leq C\Big[\mathbb{E}|u_0|_H^2 + \sum_{i=1}^k\Big(\mathbb{E}\int_0^T |a_s^i|^{\frac{\alpha_i}{\alpha_i-1}}ds\Big)^{\frac{\alpha_i-1}{\alpha_i}}\Big(\mathbb{E}\int_0^T |u_s|_{V_i}^{\alpha_i}ds\Big)^{\frac{1}{\alpha_i}}$$
$$+\mathbb{E}\sum_{j=1}^\infty\int_0^T|b_s^j|_H^2ds + \mathbb{E}\int_0^{T\wedge\sigma_n}\int_{\mathcal{D}^c}|\Gamma_s(z)|_H^2\nu(dz)ds\Big].$$

Finally taking $n\to\infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|u_t|_H^2 < \infty$$

which finishes the proof. □

In order to prove that the process u is the solution of equation (3.7), it remains to show that $dt\times\mathbb{P}$ almost everywhere $A^i(u)=a^i$ for $i=1,2,\dots,k$, $B^j(u)=b^j$ for all $j\in\mathbb{N}$ and $dt\times\mathbb{P}\times\nu$ almost everywhere $\gamma(u)1_{\mathcal{D}^c}=\Gamma 1_{\mathcal{D}^c}$. Recall that Ψ and ρ were given in Definition 3.2.

Theorem 3.4 (Existence of solution)**.** *If Assumptions A-3.1 to A-3.5 hold and $u_0\in L^{p_0}(\Omega;H)$, then the SEE* (3.7) *has a unique solution. Hence, using interlacing procedure, SEE* (3.3) *has a unique solution.*

Proof. Let $\psi\in\cap_{i=1}^k L^{\alpha_i}((0,T)\times\Omega;V_i)\cap\Psi\cap L^2(\Omega;D([0,T];H))$. Then using the product rule and Itô's formula, we obtain

$$\begin{aligned}\mathbb{E}\big(e^{-\int_0^t\rho(\psi_s)ds}|u_t|_H^2\big)-\mathbb{E}(|u_0|_H^2)=\mathbb{E}\Big[\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^k\langle a_s^i,u_s\rangle_i\\ +\sum_{j=1}^\infty|b_s^j|_H^2+\int_{\mathcal{D}^c}|\Gamma_s(z)|_H^2\nu(dz)-\rho(\psi_s)|u_s|_H^2\Big)ds\Big]\end{aligned}\tag{3.47}$$

and

$$\mathbb{E}\big(e^{-\int_0^t\rho(\psi_s)ds}|u_t^{m_q}|_H^2\big)-\mathbb{E}(|u_0^{m_q}|_H^2)=\mathbb{E}\Big[\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^k\langle A_s^i(u_s^{m_q}),u_s^{m_q}\rangle_i$$
$$+\sum_{j=1}^{m_q}|B_s^j(u_s^{m_q})|_H^2+\int_{\mathcal{D}^c}|\gamma_s(u_s^{m_q},z)|_H^2\nu(dz)-\rho(\psi_s)|u_s^{m_q}|_H^2\Big)ds\Big]$$

for all $t\in[0,T]$. Note that in view of Remark 3.4, all the integrals are well defined in what follows. Moreover,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^k\langle A_s^i(u_s^{m_q}),u_s^{m_q}\rangle_i+\sum_{j=1}^{m_q}|B_s^j(u_s^{m_q})|_H^2\\ &+\int_{\mathcal{D}^c}|\gamma_s(u_s^{m_q},z)|_H^2\nu(dz)-\rho(\psi_s)|u_s^{m_q}|_H^2\Big)ds\Big]\\ =&\mathbb{E}\Big[\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^k\langle A_s^i(u_s^{m_q})-A_s^i(\psi_s),u_s^{m_q}-\psi_s\rangle_i+2\sum_{i=1}^k\langle A_s^i(u_s^{m_q})-A_s^i(\psi_s),\psi_s\rangle_i\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i=1}^{k}\langle A_s^i(\psi_s),u_s^{m_q}\rangle_i+\sum_{j=1}^{m_q}\big|B_s^j(u_s^{m_q})-B_s^j(\psi_s)\big|_H^2+2\sum_{j=1}^{m_q}\big(B_s^j(u_s^{m_q}),B_s^j(\psi_s)\big)\\
&-\sum_{j=1}^{m_q}|B_s^j(\psi_s)|_H^2+\int_{\mathcal{D}^c}|\gamma_s(u_s^{m_q},z)-\gamma_s(\psi_s,z)|_H^2\nu(dz)+2\int_{\mathcal{D}^c}(\gamma_s(u_s^{m_q},z),\gamma_s(\psi_s,z))\nu(dz)\\
&-\int_{\mathcal{D}^c}|\gamma_s(\psi_s,z)|_H^2\nu(dz)-\rho(\psi_s)\left[|u_s^{m_q}-\psi_s|_H^2-|\psi_s|_H^2+2(u_s^{m_q},\psi_s)\right]\Big)ds\Big].
\end{aligned}
$$

Now applying the local monotonicity Assumption A-3.2, we see that

$$
\begin{aligned}
&\mathbb{E}\big(e^{-\int_0^t\rho(\psi_s)ds}|u_t^{m_q}|_H^2\big)-\mathbb{E}(|u_0^{m_q}|_H^2)\\
\leq&\mathbb{E}\Big[\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^{k}\langle A_s^i(\psi_s),u_s^{m_q}\rangle_i+2\sum_{i=1}^{k}\langle A_s^i(u_s^{m_q})-A_s^i(\psi_s),\psi_s\rangle_i\\
&-\sum_{j=1}^{m_q}|B_s^j(\psi_s)|_H^2+2\sum_{j=1}^{m_q}\big(B_s^j(u_s^{m_q}),B_s^j(\psi_s)\big)-\int_{\mathcal{D}^c}|\gamma_s(\psi_s,z)|_H^2\nu(dz)\\
&+2\int_{\mathcal{D}^c}(\gamma_s(u_s^{m_q},z),\gamma_s(\psi_s,z))\nu(dz)+\rho(\psi_s)\big[|\psi_s|_H^2-2(u_s^{m_q},\psi_s)\big]\Big)ds\Big].
\end{aligned}
$$

Integrating over t from 0 to T, letting $q\to\infty$ and using the weak lower semicontinuity of the norm we obtain,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T\big(e^{-\int_0^t\rho(\psi_s)ds}|u_t|_H^2-|u_0|_H^2\big)dt\Big]\\
&\leq\liminf_{k\to\infty}\mathbb{E}\Big[\int_0^T\big(e^{-\int_0^t\rho(\psi_s)ds}|u_t^{m_q}|_H^2-|u_0^{m_q}|_H^2\big)dt\Big]\\
&\leq\mathbb{E}\Big[\int_0^T\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^{k}\langle A_s^i(\psi_s),u_s\rangle_i+2\sum_{i=1}^{k}\langle a_s^i-A_s^i(\psi_s),\psi_s\rangle_i\\
&\quad-\sum_{j=1}^{\infty}|B_s^j(\psi_s)|_H^2+2\sum_{j=1}^{\infty}(b_s^j,B_s^j(\psi_s))-\int_{\mathcal{D}^c}|\gamma_s(\psi_s,z)|_H^2\nu(dz)\\
&\quad+2\int_{\mathcal{D}^c}(\Gamma_s(z),\gamma_s(\psi_s,z))\nu(dz)+\rho(\psi_s)\left[|\psi_s|_H^2-2(u_s,\psi_s)\right]\Big)dsdt\Big].
\end{aligned}
\tag{3.48}
$$

Integrating from 0 to T in (3.47) and combining this with (3.48) leads to,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T\int_0^t e^{-\int_0^s\rho(\psi_r)dr}\Big(2\sum_{i=1}^{k}\langle a_s^i-A_s^i(\psi_s),u_s-\psi_s\rangle_i+\sum_{j=1}^{\infty}|B_s^j(\psi_s)-b_s^j|_H^2\\
&\quad+\int_{\mathcal{D}^c}|\gamma_s(\psi_s,z)-\Gamma_s(z)|_H^2\nu(dz)-\rho(\psi_s)|u_s-\psi_s|_H^2\Big)dsdt\Big]\leq 0.
\end{aligned}
\tag{3.49}
$$

Further, using Definition 3.2, Lemma 3.2 and Lemma 3.3

$$
u\in\cap_{i=1}^k L^{\alpha_i}((0,T)\times\Omega;V_i)\cap\Psi\cap L^2(\Omega;D([0,T];H)).
$$

Taking $\psi=u$ in (3.49), we obtain that $B^j(u)=b^j$ for all $j\in\mathbb{N}$ and $\gamma(u)1_{\mathcal{D}^c}=\Gamma 1_{\mathcal{D}^c}$. Let $\eta\in L^\infty((0,T)\times\Omega;\mathbb{R})$, $\phi\in V$, $\epsilon\in(0,1)$ and let $\psi=u-\epsilon\eta\phi$. Then from (3.49) we obtain that,

$$
\mathbb{E}\Big[\int_0^T\int_0^t e^{-\int_0^s\rho(u_r-\epsilon\eta_r\phi)dr}\Big(2\epsilon\sum_{i=1}^{k}\langle a_s^i-A_s^i(u_s-\epsilon\eta_s\phi),\eta_s\phi\rangle_i-\epsilon^2\rho(u_s-\epsilon\eta_s\phi)|\eta_s\phi|_H^2\Big)dsdt\Big]\leq 0.
$$

Now we divide by ϵ and let $\epsilon\to0$. Then, using Lebesgue dominated convergence theorem and

Assumption A-3.1 we get,

$$\mathbb{E}\Big[\int_0^T\int_0^t e^{-\int_0^s \rho(u_r)dr}2\eta_s\sum_{i=1}^k\langle a_s^i - A_s^i(u_s),\phi\rangle_i dsdt\Big]\le 0.$$

Since this holds for any $\eta \in L^\infty((0,T)\times\Omega;\mathbb{R})$ and $\phi\in V$, we get that $A^i(u)=a^i$ for all $i=1,2,\ldots,k$ which concludes the proof. $\square$

3.3 Stochastic anisotropic p-Laplace equation

In this section, we prove the well-posedness of the stochastic anisotropic p-Laplace equation (3.1) in the space

$$W_0^{1,\mathbf{p}}(\mathscr{D}) := \{u|u\in L^2(\mathscr{D}), D_iu\in L^{p_i}(\mathscr{D}),\, i=1,2,\ldots,d; u=0 \text{ on } \partial\mathscr{D}\} \tag{3.50}$$

under suitable assumptions. The result is formulated in Theorem 3.5 and the proof follows by showing that (3.1), in its weak form, fits in the abstract framework discussed in previous section and hence possesses a unique solution which depends continuously on the initial data. We now describe the result in detail.

Let $\mathscr{D}\subseteq\mathbb{R}^d$ be an open bounded domain with smooth boundary. For $p_i\ge 2$, consider the spaces

$$W^{x_i,p_i}(\mathscr{D}) := \{u|u\in L^2(\mathscr{D}), D_iu\in L^{p_i}(\mathscr{D})\}.$$

It is then easy to check that the space $W^{x_i,p_i}(\mathscr{D})$ with the norm

$$|u|_{i,p_i} := |u|_{L^2} + [u]_{i,p_i}$$

is a Banach space, where $[u]_{i,p_i} := |D_iu|_{L^{p_i}}$ is a semi-norm. Let $W_0^{x_i,p_i}(\mathscr{D})$ be the closure of $C_0^\infty(\mathscr{D})$ in $W^{x_i,p_i}(\mathscr{D})$. It can be seen that each $W_0^{x_i,p_i}(\mathscr{D})$ is a separable and reflexive Banach space and $W_0^{1,\mathbf{p}}(\mathscr{D}) = \cap_{i=1}^d W_0^{x_i,p_i}(\mathscr{D})$ is embedded continuously and densely in the space $L^2(\mathscr{D})$.

Assume that $\gamma:[0,T]\times\Omega\times W_0^{1,\mathbf{p}}(\mathscr{D})\times Z\to L^2(\mathscr{D})$ is a $\mathscr{P}\times\mathscr{B}(W_0^{1,\mathbf{p}})\times\mathscr{Z}$-measurable function. Finally, u_0 is assumed to be a given $L^2(\mathscr{D})$-valued, $\mathscr{F}_0$-measurable random variable.

Definition 3.3 (Solution)**.** An adapted, càdlàg, $L^2(\mathscr{D})$-valued process u is called a solution of the stochastic anisotropic p-Laplace equation (3.1) if

i) $dt\times\mathbb{P}$ almost everywhere $u\in W_0^{1,\mathbf{p}}(\mathscr{D})$ and

$$\mathbb{E}\int_0^T\int_{\mathscr{D}}\Big(|u_t(x)|^2+\sum_{i=1}^d|D_iu_t(x)|^{p_i}\Big)dxdt<\infty\,,$$

ii) for every $t\in[0,T]$ and $\phi\in W_0^{1,\mathbf{p}}(\mathscr{D})$,

$$\begin{aligned}\int_{\mathscr{D}}u_t(x)\phi(x)dx &= \int_{\mathscr{D}}u_0(x)\phi(x)dx-\sum_{i=1}^d\int_0^t\int_{\mathscr{D}}|D_iu_s|^{p_i-2}D_iu_s(x)D_i\phi(x)dxds\\ &+\sum_{j=1}^d\int_0^t\int_{\mathscr{D}}\zeta_j|D_ju_s(x)|^{\frac{p_j}{2}}\phi(x)dxdW_s^j+\sum_{j=1}^\infty\int_0^t\int_{\mathscr{D}}h_j(u_s(x))\phi(x)dxdW_s^j\\ &+\int_0^t\int_{\mathcal{D}^c}\int_{\mathscr{D}}\phi(x)\gamma_s(u_s(x),z)dx\tilde{N}(ds,dz)+\int_0^t\int_{\mathcal{D}}\int_{\mathscr{D}}\phi(x)\gamma_s(u_s(x),z)dxN(ds,dz)\end{aligned}$$

almost surely.

We now formulate the result regarding well-posedness of stochastic anisotropic p-Laplace equation (3.1).

Theorem 3.5. *Assume that there exists constants $p_0\ge\max\{p_1,p_2,\ldots,p_d\}$, $\zeta_j^2\le\frac{2(p_j-1)}{p_j^2(p_0-1)}$ and $K>0$ such that almost surely, the following conditions hold for all $t\in[0,T]$.*

1. *For all* $u, v \in W_0^{1,\mathbf{p}}(\mathscr{D})$,

$$\int_{\mathcal{D}^c}\int_{\mathscr{D}} |\gamma_t(u,z)-\gamma_t(v,z)|^2\,dx\nu(dz) \le K\int_{\mathscr{D}} |u-v|^2\,dx \tag{3.51}$$

2. *For all* $u \in W_0^{1,\mathbf{p}}(\mathscr{D})$,

$$\int_{\mathcal{D}^c}\int_{\mathscr{D}} |\gamma_t(u,z)|^2\,dx\nu(dz) \le K\Big(1+\int_{\mathscr{D}} |u|^2\,dx\Big) \tag{3.52}$$

and

$$\int_{\mathcal{D}^c}\Big(\int_{\mathscr{D}} |\gamma_t(u,z)|^2\,dx\Big)^{\frac{p_0}{2}}\nu(dz) \le K\Big(1+\Big(\int_{\mathscr{D}} |u|^2\,dx\Big)^{\frac{p_0}{2}}\Big). \tag{3.53}$$

Further, if the initial condition $u_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$ *and* $h_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}$ *are Lipschitz continuous functions with Lipschitz constants* M_j *such that the sequence* $(M_j)_{j\in\mathbb{N}} \in \ell^2$, *then there exists a unique solution of anisotropic p-Laplace equation* (3.1) *in the sense of Definition 3.3. Furthermore, if* u *and* $\bar{u}$ *are two solutions with initial condition* u_0 *and* $\bar{u}_0$ *respectively,* $u_0, \bar{u}_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$ *and for all* $u, v \in W_0^{1,\mathbf{p}}(\mathscr{D})$,

$$\int_{\mathcal{D}^c}\Big(\int_{\mathscr{D}} |\gamma_t(u,z)-\gamma_t(v,z)|^2\Big)^{\frac{p_0}{2}}\,dx\nu(dz) \le K\Big(\int_{\mathscr{D}} |u-v|^2\,dx\Big)^{\frac{p_0}{2}}, \tag{3.54}$$

then,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |u_t-\bar{u}_t|^p_{L^2} + \sum_{i=1}^d \int_0^T |D_iu_t - D_i\bar{u}_t|^{p_i}_{L^{p_i}}\,dt\Big) < C\mathbb{E}|u_0-\bar{u}_0|^{p_0}_{L^2} \tag{3.55}$$

with $p = 2$ *in case* $p_0 = 2$ *and with any* $p \in [2, p_0)$ *in case* $p_0 > 2$.

Proof. For $i = 1, 2, \ldots, d$, take $V_i := W_0^{x_i,p_i}(\mathscr{D})$ defined above so that the space V is the space $W_0^{1,\mathbf{p}}(\mathscr{D})$ given by (3.50). Again for $i = 1, 2, \ldots, d$, let $A^i : V_i \to V_i^*$ be given by,

$$A^i(u) := D_i\big(|D_iu|^{p_i-2}D_iu\big)\,.$$

Further, let $B^j : V \to L^2(\mathscr{D})$ be given by,

$$B^j(u) := \begin{cases} \zeta_j|D_ju|^{\frac{p_j}{2}} + h_j(u) & \text{for } j = 1, 2, \ldots, d, \\ h_j(u) & \text{otherwise.} \end{cases}$$

We note that for $u, v \in V_i$,

$$\langle A_i(u), v\rangle_i = -\int_{\mathscr{D}} |D_iu(x)|^{p_i-2}D_iu(x)D_iv(x)dx \tag{3.56}$$

and thus using Hölder's inequality,

$$\big|\langle A_i(u), v\rangle_i\big| \le |u|^{p_i-1}_{V_i}|v|_{V_i}\,.$$

Thus, for every $u \in V^i$, $A^i(u)$ is a well-defined linear operator on V_i such that

$$|A_iu|_{V_i^*} \le |u|^{p_i-1}_{V_i}$$

which implies that Assumptions A-3.1 and A-3.4 hold with $\alpha_i = p_i$ and $\beta = 0$.

We now verify the local monotonicity condition. As observed in Example 2.4, we obtain for each $i = 1, 2, \ldots, d$,

$$\big\langle D_i\big(|D_iu|^{p_i-2}D_iu\big) - D_i\big(|D_iv|^{p_i-2}D_iv\big), u-v\big\rangle_i + \big|\zeta_i|D_iu|^{\frac{p_i}{2}} - \zeta_i|D_iv|^{\frac{p_i}{2}}\big|^2_{L^2} \le 0$$

provided $\zeta_i^2 \le \frac{4(p_i-1)}{p_i^2}$. Since the functions h_j, $j \in \mathbb{N}$ are given to be Lipschitz continuous with

Lipschitz constants M_j such that $(M_j)_{j\in\mathbb{N}} \in \ell^2$, we have

$$|h_j(u) - h_j(v)|_{L^2}^2 \leq M_j^2 |u-v|_{L^2}^2 \,. \tag{3.57}$$

Using (3.51), we get

$$\int_{\mathcal{D}^c} |\gamma(u,z) - \gamma(v,z)|_{L^2}^2 \, \nu(dz) \leq K|u-v|_{L^2}^2 \,. \tag{3.58}$$

Therefore,

$$2\sum_{i=1}^{d} \langle A^i(u) - A^i(v), u-v\rangle_i + \sum_{j=1}^{\infty} |B^j(u) - B^j(v)|_{L^2}^2 + \int_{\mathcal{D}^c} |\gamma(u,z) - \gamma(v,z)|_{L^2}^2 \nu(dz)$$
$$\leq C|u-v|_{L^2}^2$$

and hence Assumption A-3.2 is satisfied.

We now wish to verify the p_0-stochastic coercivity condition A-3.3. However, in view of Remark 3.3, it is enough to verify Assumption A-3.6 instead. Taking $v = u$ in (3.56), we get

$$\langle A^i(u), u\rangle_i = -\int_{\mathscr{D}} |D_i u(x)|^{p_i} dx \,.$$

Further,

$$2(p_0 - 1)\big|\zeta_i |D_i u|^{\frac{p_i}{2}}\big|_{L^2}^2 = 2(p_0-1)\zeta_i^2 \int_{\mathscr{D}} |D_i u(x)|^{p_i} dx.$$

Also, (3.52) gives

$$\int_{\mathcal{D}^c} |\gamma(u,z)|_{L^2}^2 \, \nu(dz) \leq K(1 + |u|_{L^2}^2) \,.$$

Choose $\zeta_i^2 < \frac{1}{(p_0-1)}$, so that $\theta_i := 2 - 2(p_0-1)\zeta_i^2 > 0$. Then taking θ to be the minimum of $\theta_1, \theta_2, \ldots, \theta_d$ we have,

$$2\sum_{i=1}^{d} \langle A^i(u), u\rangle_i + (p_0 - 1)\sum_{i=1}^{\infty} |B^i(u)|_{L^2}^2 + \theta \sum_{i=1}^{d} [u]_{V_i}^{p_i} + \int_{\mathcal{D}^c} |\gamma(u,z)|_{L^2}^2 \nu(dz)$$
$$\leq C(1 + |u|_{L^2}^2)$$

where, $[u]_{V_i}^{p_i} := \int_{\mathscr{D}} |D_i u(x)|^{p_i} dx$ and thus Assumption A-3.6 is satisfied. Finally, we need to verify Assumption A-3.5. Using (3.53), we have

$$\int_{\mathcal{D}^c} |\gamma(u,z)|_{L^2}^{p_0} \, \nu(dz) \leq K(1 + |u|_{L^2}^{p_0})$$

as desired. Since $u_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$, in view of Remark 3.3 along with Theorems 3.1, 3.2 and 3.4, stochastic anisotropic p-Laplace equation (3.1) has a unique solution.

We now show the continuous dependence of the solution on the initial data by proving (3.55). For this, we show that operators in (3.1) satisfy Assumptions A-3.7 to A-3.9. Using the inequality

$$(|a|^r a - |b|^r b)(a-b) \geq 2^{-r}|a-b|^{r+2} \quad \forall\, r \geq 0,\, a, b \in \mathbb{R},$$

we have for each $i = 1, 2, \ldots, d$,

$$\big\langle D_i\big(|D_i u|^{p_i - 2} D_i u\big) - D_i\big(|D_i v|^{p_i-2} D_i v\big), u - v\big\rangle_i \leq -2^{-(p_i-2)} |D_i u - D_i v|_{L^{p_i}}^{p_i} \,.$$

Further as discussed above,

$$\big\langle D_i\big(|D_i u|^{p_i-2} D_i u\big) - D_i\big(|D_i v|^{p_i-2} D_i v\big), u-v\big\rangle_i + 2(p_0-1)\big|\zeta_i |D_i u|^{\frac{p_i}{2}} - \zeta_i |D_i v|^{\frac{p_i}{2}}\big|_{L^2}^2 \leq 0$$

provided $\zeta_i^2 \leq \frac{2(p_i-1)}{p_i^2(p_0-1)}$. Thus we have for $u, v \in W_0^{1,\mathbf{p}}(\mathscr{D})$,

$$
\begin{aligned}
& 2\sum_{i=1}^{d}\langle A^i(u)-A^i(v), u-v\rangle_i + (p_0-1)\sum_{j=1}^{\infty}|B^j(u)-B^j(v)|_{L^2}^2 \\
& + \int_{\mathcal{D}^c}|\gamma(u,z)-\gamma(v,z)|_{L^2}^2\nu(dz) \leq -\theta'\sum_{i=1}^{d}|D_iu-D_iv|_{L^{p_i}}^{p_i} + C|u-v|_{L^2}^2
\end{aligned}
$$

for any θ' satisfying $0 < \theta' < 2^{-(p_i-2)}$ for all i. Thus, Assumption A-3.7 is satisfied. Using (3.54), we obtain

$$
\int_{\mathcal{D}^c}|\gamma_t(u,z)-\gamma_t(v,z)|_{L^2}^{p_0}\nu(dz) \leq K|u-v|_{L^2}^{p_0}
$$

showing that Assumption A-3.8 holds. Finally, for each $i = 1, 2, \ldots, d$,

$$
\left|\zeta_i|D_iu|^{\frac{p_i}{2}} - \zeta_i|D_iv|^{\frac{p_i}{2}}\right|_{L^2}^2 \leq C\left||D_iu-D_iv|^{\frac{p_i}{2}}\right|_{L^2}^2 \leq C|u-v|_{V_i}^{p_i}
$$

and therefore using (3.57) and (3.58), we obtain

$$
\sum_{j=1}^{\infty}|B_t^j(u)-B_t^j(v)|_{L^2}^2 + \int_{\mathcal{D}^c}|\gamma_t(u,z)-\gamma_t(v,z)|_{L^2}^2\nu(dz) \leq K\Big(|u-v|_{L^2}^2 + \sum_{i=1}^{d}|u-v|_{V_i}^{p_i}\Big)
$$

and hence (3.55) follows from Theorem 3.3 and Remark 3.5. This concludes the proof of Theorem 3.5 and hence establishes the well-posedness of stochastic anisotropic p-Laplace equation (3.1). □

3.4 Example

Finally, in this section, we present an example of stochastic evolution equation which fits in the framework of this chapter and yet does not satisfy the assumptions of [3, 28] or [30]. Here, $\mathscr{D} \subseteq \mathbb{R}^d$ is an open bounded domain with smooth boundary and $W_0^{1,p}(\mathscr{D})$ is the Sobolev space defined in Chapter 1.

Example 3.1 (Quasi-linear equation). Let $p_1, p_2 > 2$. Assume that there are constants $r, s, t \geq 1$ and continuous function f^0 on $\mathbb{R}$ such that

$$
\begin{aligned}
f^0(x)x \leq K(1+|x|^{\frac{p_1}{2}+1});\ \ |f^0(x)| \leq K(1+|x|^r) \\
\text{and}\ \ (f^0(x)-f^0(y))(x-y) \leq K(1+|y|^s)|x-y|^t\ \ \forall\ x, y \in \mathbb{R}\,.
\end{aligned}
$$

Let $h_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}$ be Lipschitz continuous functions with Lipschitz constants M_j such that the sequence $(M_j)_{j\in\mathbb{N}} \in \ell^2$. Further, let $Z = \mathbb{R}^d$, $\mathcal{D}^c = \{z \in \mathbb{R}^d : |z| \leq 1\}$ and ν be a Lévy measure on $\mathbb{R}^d$. Finally assume that $\gamma : [0,T]\times\Omega\times\mathbb{R}\times Z \to Z$ satisfies

$$
|\gamma_t(x,z)-\gamma_t(y,z)| \leq K|x-y||z| \ \text{ and } \ |\gamma_t(x,z)| \leq K(1+|x|)|z|
$$

almost surely, for all $t \in [0,T]$, $x, y \in \mathbb{R}$, $z \in \mathcal{D}^c$.

Consider the stochastic partial differential equation,

$$
\begin{aligned}
du_t =& \Big(\sum_{\ell=1}^{d}D_\ell\big(|D_\ell u_t|^{p_1-2}D_\ell u_t\big) - |u_t|^{p_2-2}u_t + f^0(u_t)\Big)\,dt + \sum_{j=1}^{d}\zeta|D_ju_t|^{\frac{p_1}{2}}dW_t^j \\
& + \sum_{j=1}^{\infty}h_j(u_t)dW_t^j + \int_{\mathcal{D}^c}\gamma_t(u_t,z)\tilde{N}(dt,dz) + \int_{\mathcal{D}}\gamma_t(u_t,z)N(dt,dz)
\end{aligned}
\tag{3.59}
$$

on $(0,T)\times\mathscr{D}$, where $u_t = 0$ on $\partial\mathscr{D}$, u_0 is a given $\mathcal{F}_0$-measurable random variable and ζ is a constant to be chosen suitably. Moreover, W^j are independent Wiener processes. We will now show that such an equation, in its weak form, fits the assumptions made in this chapter if any of the following holds:

1. $d < p_1$, $r = p_1 + 1$, $s \le p_1$, $t = 2$ and $u_0 \in L^6(\Omega; L^2(\mathscr{D}))$.

2. $d > p_1$, $r = \frac{2p_1}{d} + p_1 - 1$, $s \le \min\left\{\frac{p_1^2(t-2)}{(d-p_1)(p_1-2)}, \frac{p_1(p_1-t)}{(p_1-2)}\right\}$, $2 < t < p_1$ and $u_0 \in L^6(\Omega; L^2(\mathscr{D}))$.

Case 1. Take $V_1 := W_0^{1,p_1}(\mathscr{D})$, $V_2 := L^{p_2}(\mathscr{D})$ and $V := V_1 \cap V_2$. Then $(V_i, |\cdot|_{V_i})$ are reflexive and separable Banach spaces such that

$$V \hookrightarrow L^2(\mathscr{D}) \equiv (L^2(\mathscr{D}))^* \hookrightarrow V^*.$$

Let $A^1 : V_1 \to V_1^*$ and $A^2 : V_2 \to V_2^*$ be given by,

$$A^1(u) := \sum_{l=1}^{d} D_l\big(|D_l u|^{p_1-2} D_l u\big) + f^0(u) \text{ and } A^2(u) := -|u|^{p_2-2} u\,.$$

Moreover, $B^j : V \to L^2(\mathscr{D})$ be given by

$$B^j(u) := \begin{cases} \zeta |D_j u|^{\frac{p_1}{2}} + h_j(u) & \text{for } j = 1, 2, \ldots, d, \\ h_j(u) & \text{otherwise}\,. \end{cases}$$

The next step is to show that these operators satisfy the Assumptions A-3.1 to A-3.5. We immediately notice that A-3.1 holds since f^0 is continuous.

We now wish to verify the local monotonicity condition. As discussed earlier in Example 2.4, for each $l = 1, 2, \ldots, d$

$$\big\langle D_l\big(|D_l u|^{p_1-2} D_l u\big) - D_l\big(|D_l v|^{p_1-2} D_l v\big), u - v\big\rangle_1 + \big|\zeta |D_l u|^{\frac{p_1}{2}} - \zeta |D_l v|^{\frac{p_1}{2}}\big|_{L^2}^2 \le 0$$

provided $\zeta^2 \le \frac{4(p_1-1)}{p_1^2}$. Since the function $-|x|^{p_2-2}x$ is monotonically decreasing, we get

$$\langle -|u|^{p_2-2} u + |v|^{p_2-2} v, u - v\rangle_2 \le 0.$$

Further for $d < p_1$, by Sobolev embedding we have $V_1 \subset L^\infty(\mathscr{D})$ and therefore using the assumptions imposed on f_0 taking $t = 2$, we observe that for $u, v \in V$

$$\begin{aligned} \langle f^0(u) - f^0(v), u - v\rangle_1 &\le K \int_{\mathscr{D}} (1 + |v(x)|^s)|u(x) - v(x)|^2 dx \\ &\le K(1 + |v|_{L^\infty}^s)|u - v|_{L^2}^2 \le C(1 + |v|_{V_1}^{p_1})|u - v|_{L^2}^2 \end{aligned}$$

for $s \le p_1$. Using Lipschitz continuity of the functions h_j, $j \in \mathbb{N}$, we have

$$|h_j(u) - h_j(v)|_{L^2}^2 \le M_j^2 |u - v|_{L^2}^2\,,$$

where M_j are the Lipschitz constants such that $(M_j)_{j\in\mathbb{N}} \in \ell^2$. Again using assumptions imposed on γ and the fact that ν is a Lévy measure, we have

$$\begin{aligned} \int_{\mathcal{D}^c} |\gamma(u, z) - \gamma(v, z)|_{L^2}^2 \nu(dz) &\le \int_{\mathcal{D}^c} \int_{\mathscr{D}} |u(x) - v(x)|^2 |z|^2 dx \nu(dz) \\ &= K \int_{\mathcal{D}^c} |z|^2 \nu(dz) \int_{\mathscr{D}} |u(x) - v(x)|^2 dx \le C|u - v|_{L^2}^2\,. \end{aligned}$$

Therefore, we have for all $u, v \in V$

$$2\sum_{i=1}^{2} \langle A^i(u) - A^i(v), u - v\rangle_i + \sum_{j=1}^{\infty} |B^j(u) - B^j(v)|_{L^2}^2 + \int_{\mathcal{D}^c} |\gamma(u, z) - \gamma(v, z)|_{L^2}^2 \nu(dz)$$

$$\leq C\Big(1+|v|_{V_1}^{p_1}\Big)|u-v|_{L^2}^2 \leq C\Big(1+\sum_{i=1}^2 |v|_{V_i}^{p_i}\Big)|u-v|_{L^2}^2.$$

Hence Assumption A-3.2 is satisfied with $\alpha_i := p_i$ $(i=1,2)$ and $\beta := 0$. Again,

$$2\sum_{l=1}^d \big\langle D_l\big(|D_l u|^{p_1-2}D_l u\big), u\big\rangle_1 = -2\sum_{l=1}^d \int_{\mathscr{D}} |D_l u(x)|^{p_1}dx = -2|u|_{V_1}^{p_1}$$

and similarly,

$$2\langle -|u|^{p_2-2}u, u\rangle_2 = -2|u|_{V_2}^{p_2}.$$

Moreover using assumptions on f^0 and Sobolev embedding, we get

$$\begin{aligned}2\langle f^0(u), u\rangle_1 &\leq K\int_{\mathscr{D}}(1+|u(x)|^{\frac{p_1}{2}+1})dx\\ &\leq K(1+|u|_{L^\infty}^{\frac{p_1}{2}}|u|_{L^2}) \leq C(1+|u|_{V_1}^{\frac{p_1}{2}}|u|_{L^2}) \leq \delta|u|_{V_1}^{p_1}+C(1+|u|_{L^2}^2),\end{aligned}$$

where last inequality is obtained using Young's inequality with sufficiently small $\delta>0$. Further, for any $p_0>2$

$$2(p_0-1)\sum_{j=1}^d \big|\zeta|D_j u|^{\frac{p_1}{2}}\big|_{L^2}^2 = 2(p_0-1)\zeta^2\sum_{j=1}^d\int_{\mathscr{D}}|D_j u(x)|^{p_1}dx = 2(p_0-1)\zeta^2|u|_{V_1}^{p_1}.$$

Furthermore, using assumptions on γ and the fact that ν is a Lévy measure on $\mathbb{R}^d$, we get

$$\begin{aligned}\int_{\mathcal{D}^c}|\gamma(u,z)|_{L^2}^2\nu(dz) &\leq K\int_{\mathcal{D}^c}\int_{\mathscr{D}}\big(1+|u(x)|\big)^2|z|^2dx\nu(dz)\\ &= K\int_{\mathcal{D}^c}|z|^2\nu(dz)\int_{\mathscr{D}}\big(1+|u(x)|^2\big)dx \leq C\big(1+|u|_{L^2}^2\big).\end{aligned}$$

Choose $\zeta^2 < \frac{2-\delta}{2(p_0-1)}$, so that $\theta := 2-2(p_0-1)\zeta^2-\delta>0$. Then we have,

$$2\sum_{i=1}^2\langle A^i(u),u\rangle_i + (p_0-1)\sum_{j=1}^\infty |B^j(u)|_{L^2}^2 + \theta\sum_{i=1}^d |u|_{V_i}^{p_i} + \int_{\mathcal{D}^c}|\gamma(u,z)|_{L^2}^2\nu(dz) \leq C(1+|u|_{L^2}^2).$$

Hence Assumption A-2.3 is satisfied with $\alpha_i := p_i$ $(i=1,2)$. Again, using the assumptions on γ, we have

$$\begin{aligned}\int_{\mathcal{D}^c}|\gamma(u,z)|_{L^2}^{p_0}\nu(dz) &= \int_{\mathcal{D}^c}\Big(\int_{\mathscr{D}}|\gamma(u(x),z)|^2dx\Big)^{\frac{p_0}{2}}\nu(dz)\\ &\leq K\int_{\mathcal{D}^c}\Big(\int_{\mathscr{D}}\big(1+|u(x)|\big)^2|z|^2dx\Big)^{\frac{p_0}{2}}\nu(dz)\\ &= K\int_{\mathcal{D}^c}|z|^{p_0}\nu(dz)\Big(\int_{\mathscr{D}}\big(1+|u(x)|^2\big)dx\Big)^{\frac{p_0}{2}}\\ &\leq C\int_{\mathcal{D}^c}|z|^2\nu(dz)\Big[1+\Big(\int_{\mathscr{D}}|u(x)|^2dx\Big)^{\frac{p_0}{2}}\Big] \leq C(1+|u|_{L^2}^{p_0})\end{aligned}$$

and hence Assumption A-3.5 is satisfied. Note that using Hölder's inequality, we get for $u,v\in V_1$

$$\begin{aligned}\int_{\mathscr{D}}|D_l u(x)|^{p_1-1}|D_l v(x)|dx &\leq \Big(\int_{\mathscr{D}}|D_l u(x)|^{p_1}dx\Big)^{\frac{p_1-1}{p_1}}\Big(\int_{\mathscr{D}}|D_l v(x)|^{p_1}dx\Big)^{\frac{1}{p_1}}\\ &\leq \Big(\sum_{l=1}^d\int_{\mathscr{D}}|D_l u(x)|^{p_1}dx\Big)^{\frac{p_1-1}{p_1}}\Big(\sum_{l=1}^d\int_{\mathscr{D}}|D_l v(x)|^{p_1}dx\Big)^{\frac{1}{p_1}}\\ &= |u|_{V_1}^{p_1-1}|v|_{V_1}.\end{aligned}$$

Further using assumption on f^0 taking $r = p_1 + 1$ and Sobolev embedding, we have for $u, v \in V_1$

$$\begin{aligned}\int_{\mathscr{D}} |f^0(u(x))||v(x)|dx &\le K \int_{\mathscr{D}} \big(1 + |u(x)|^{p_1+1}\big)|v(x)|dx \\ &\le K|v|_{L^2} + K|v|_{L^\infty}|u|^{p_1+1}_{L^{p_1+1}} \\ &\le K|v|_{V_1}(1 + |u|^{p_1-1}_{L^\infty}|u|^2_{L^2}) \\ &\le K|v|_{V_1}(1 + |u|^{p_1-1}_{V_1}|u|^2_{L^2})\end{aligned}$$

and hence

$$|A^1(u)|_{V_1^*} \le K|u|^{p_1-1}_{V_1} + K(1 + |u|^{p_1-1}_{V_1}|u|^2_{L^2}) \le K(1 + |u|^{p_1-1}_{V_1})(1 + |u|^2_{L^2})\,.$$

Again, using Hölder's inequality

$$|A^2(u)|_{V_2^*} \le K|u|^{p_2-1}_{V_2}\,,$$

which implies that Assumption A-3.4 holds with $\alpha_i := p_i$ $(i = 1, 2)$ and $\beta = \frac{2p_1}{p_1-1} < 4$. Thus taking $p_0 = 6$ and $u_0 \in L^6(\Omega; L^2(\mathscr{D}))$, in view of Theorems 3.1, 3.2 and 3.4, equation (3.59) has a unique solution and moreover for any $p < 6$ we have,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |u_t|^p_{L^2} + \sum_{i=1}^{2}\int_0^T |u_t|^{\alpha_i}_{V_i}dt\Big) < C\left(1 + \mathbb{E}|u_0|^6_{L^2}\right).$$

Case 2. In the case $d > p_1$, one can obtain the result in a similar manner, as in Example 2.4, using the Sobolev embedding $W_0^{1,p_1}(\mathscr{D}) \subset L^{\frac{dp_1}{d-p_1}}(\mathscr{D})$ and interpolation inequalities stated in Example 2.4.

3.5 Interlacing procedure for SPDEs

In this section, we present how the interlacing procedure can be used to construct the unique solution of SEE (3.3) with large jumps from the unique solution of the corresponding SEE (3.7) with only small jumps. The work presented in this section is based on the interlacing procedure for a class of SDEs presented in Ikeda and Watanabe [17, Chapter 4, Section 9] and for a class of SPDEs in [3, Section 4.2]. Further we refer the reader to [17], for the details of the notions and results used in this section.

Let p be the Poisson point process associated to the Poisson random measure $N(dt, dz)$ and $\mathsf{D}(\mathsf{p})$ be its domain. It is well-known that p is stationary if and only if there exists a non-negative measure ν on $(Z, \mathscr{Z})$ such that $\mathbb{E}N((0, t] \times A) = t\nu(A)$ for all $t > 0, A \in \mathscr{Z}$. Thus it follows that p is a stationary $\mathscr{F}_t$-Poisson point process. Further, the assumption $\nu(\mathcal{D}) < \infty$ for $\mathcal{D} \in \mathscr{Z}$ implies that the set,

$$\{s \in (0, T] \cap \mathsf{D}(\mathsf{p}) : \mathsf{p}(s, \omega) \in \mathcal{D}\}$$

is finite almost surely. Note that the points in this set corresponds to the jump times of the Poisson process $N((0, t] \times \mathcal{D})$, $t \in (0, T]$. Let $\tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ be the enumeration of these points. Then $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times converging to T $\mathbb{P}$-a.s. as $n \to \infty$.

Before explaining the procedure, we recall the strong Markov property of the Brownian motions and Poisson point processes (see, e.g. [17, Chapter 2, Theorems 6.4 and 6.5]).

Lemma 3.4 (Strong Markov Property). *Let τ be a stopping time which is finite almost surely. Define,*

$$W_t^\tau = W_{t+\tau} - W_\tau\,,\ t \in [0, T - \tau]; \quad \mathsf{p}_t^\tau = \mathsf{p}_{t+\tau},\, t \in \mathsf{D}(\mathsf{p}^\tau) := \{t \in (0, \infty) : t + \tau \in \mathsf{D}(\mathsf{p})\}$$
and $\mathscr{F}_t^\tau = \mathscr{F}_{t+\tau},\, t \in [0, T - \tau]$.

Then, $(W_t^\tau)_{t\in[0,T-\tau]}$ is an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t^\tau)_{t\in[0,T-\tau]}$

and p^τ is a stationary $\mathscr{F}_t^\tau$-Poisson point process with intensity measure ν.

Note that W^τ, p^τ have the same properties as W, p. Thus, we have the following result.

Lemma 3.5. *Let τ be a stopping time taking values in $[0,T]$ and u_τ be an H-valued, $\mathscr{F}_\tau$-measurable random variable such that $u_\tau \in L^{p_0}(\Omega; H)$. Under the assumptions of Theorem 3.4, there exists a unique adapted, càdlàg, H-valued process u such that $dt \times \mathbb{P}$ almost everywhere $u \in V$. Further, $u \in \cap_{i=1}^k L^{\alpha_i}((\tau,T)\times\Omega; V_i) \cap L^{p_0}((\tau,T)\times\Omega; H)$ and almost surely,*

$$u_t = u_\tau + \sum_{i=1}^k \int_\tau^t A_s^i(u_s)ds + \sum_{j=1}^\infty \int_\tau^t B_s^j(u_s)dW_s^j + \int_\tau^t \int_{\mathcal{D}^c} \gamma_s(u_s, z)\tilde{N}(ds,dz)$$

for $t \in [\tau, T]$.

Proof. First we assume that $u_\tau = h \in H$. Clearly, $u_\tau \in L^{p_0}(\Omega; H)$. Let $\tilde{N}^\tau(dt,dz)$ be the compensated Poisson random measure associated to the Poisson point process p^τ. Using Lemma 3.4 and working along the same lines of the proof of Theorem 3.4 replacing all the computations involving the expectations by conditional expectations with respect to $\mathscr{F}_\tau$, there exists a unique $(\mathscr{F}_t^\tau)$-adapted càdlàg, H-valued process $u^{\tau,h}$ such that,

$$u_t^{\tau,h} = h + \sum_{i=1}^k \int_0^t A_{s+\tau}^i(u_s^{\tau,h})ds + \sum_{j=1}^\infty \int_0^t B_{s+\tau}^j(u_s^{\tau,h})dW_s^{\tau j} + \int_0^t \int_{\mathcal{D}^c} \gamma_{s+\tau}(u_s^{\tau,h}, z)\tilde{N}^\tau(ds,dz)$$

for $t \in [0, T-\tau]$. Since for any $h \in H$, the solution $u_t^{\tau,h}$ is a measurable function of h, the solution for a general initial value u_τ, where u_τ, W^τ and p^τ are mutually independent, is obtained by replacing h with the $\mathscr{F}_\tau$-measurable random variable u_τ. Thus, we obtain a unique $(\mathscr{F}_t^\tau)$-adapted càdlàg, H-valued process u^τ such that

$$u_t^\tau = u_\tau + \sum_{i=1}^k \int_0^t A_{s+\tau}^i(u_s^\tau)ds + \sum_{j=1}^\infty \int_0^t B_{s+\tau}^j(u_s^\tau)dW_s^{\tau j} + \int_0^t \int_{\mathcal{D}^c} \gamma_{s+\tau}(u_s^\tau, z)\tilde{N}^\tau(ds,dz)$$

for $t \in [0, T-\tau]$. Substituting $s+\tau = r$, above equation can be rewritten as

$$u_t^\tau = u_\tau + \sum_{i=1}^k \int_\tau^{t+\tau} A_r^i(u_{r-\tau}^\tau)dr + \sum_{j=1}^\infty \int_\tau^{t+\tau} B_r^j(u_{r-\tau}^\tau)dW_r^j + \int_\tau^{t+\tau} \int_{\mathcal{D}^c} \gamma_r(u_{r-\tau}^\tau, z)\tilde{N}(dr,dz)$$

for $t \in [0, T-\tau]$. Finally, defining $u_t := u_{t-\tau}^\tau$ for $t \in [\tau, T]$, we observe that u_t is the desired solution of SEE (3.7) in the interval $[\tau, T]$ with initial condition u_τ and hence the result. ☐

The unique solution of SEE (3.7) in the interval $[\tau, T]$ with initial condition u_τ, obtained using Lemma 3.5 above, will be denoted by $\tilde{u}_{\tau,t}(u_\tau)$, $t \in [\tau, T]$ in what follows. Further, we use the notation $u_{\tau,t}(u_\tau)$, $t \in [\tau, T]$ to denote the solution of SEE (3.3) in the interval $[\tau, T]$ with initial condition u_τ.

We now construct the unique solution to SEE (3.3) by using Theorem 3.4 and Lemma 3.5 repeatedly.

Recall that $(\tau_n)_{n\in\mathbb{N}}$ is the sequence of the jump times of the Poisson process $N((0,t]\times \mathcal{D})$, $t \in (0, T]$. From Theorem 3.4, there exists a unique solution $\tilde{u}_{0,t}(u_0)$ to SEE (3.7) with initial condition u_0 in the interval $[0, T]$. Thus,

$$\tilde{u}_{0,t}(u_0) = u_0 + \sum_{i=1}^k \int_0^t A_s^i(\tilde{u}_{0,s}(u_0))ds + \sum_{j=1}^\infty \int_0^t B_s^j(\tilde{u}_{0,s}(u_0))dW_s^j + \int_0^t \int_{\mathcal{D}^c} \gamma_s(\tilde{u}_{0,s}(u_0), z)\tilde{N}(ds,dz)$$

for $t \in [0, T]$. We construct a solution to SEE (3.3) on $[0, \tau_1]$ as follows:

$$u_{0,t}(u_0) = \begin{cases} \tilde{u}_{0,t}(u_0) & \text{for} \quad 0 \le t < \tau_1, \\ \tilde{u}_{0,\tau_1^-}(u_0) + \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1^-}(u_0), \mathsf{p}(\tau_1)\big) & \text{for} \quad t = \tau_1. \end{cases}$$

where we note that $u_{0,\tau_1^-}(u_0) = \tilde{u}_{0,\tau_1^-}(u_0) = \tilde{u}_{0,\tau_1}(u_0)$ as the H-valued process $\tilde{u}_{0,t}(u_0)$, $t \in [0,T]$ has no jump at time τ_1. Clearly, the processes $\tilde{u}_{0,t}(u_0)$ and $u_{0,t}(u_0)$ are equivalent $dt \times \mathbb{P}$-almost everywhere. Thus we have,

$$\begin{aligned} u_{0,\tau_1}(u_0) =& \tilde{u}_{0,\tau_1}(u_0) + \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1}(u_0), \mathsf{p}(\tau_1)\big) \\ =& u_0 + \sum_{i=1}^{k} \int_0^{\tau_1} A_s^i(\tilde{u}_{0,s}(u_0))ds + \sum_{j=1}^{\infty} \int_0^{\tau_1} B_s^j(\tilde{u}_{0,s}(u_0))dW_s^j \\ &+ \int_0^{\tau_1} \int_{\mathcal{D}^c} \gamma_s(\tilde{u}_{0,s}(u_0), z)\tilde{N}(ds,dz) + \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1}(u_0), \mathsf{p}(\tau_1)\big). \end{aligned}$$

Since τ_1 is the first arrival time for the jump of the Poisson process $N((0,t] \times \mathcal{D})$, $t \in (0,T]$, we have

$$\int_0^t \int_{\mathcal{D}} \gamma_s(\tilde{u}_{0,s}(u_0), z)N(ds,dz) = \begin{cases} 0 & \text{for} \quad 0 \le t < \tau_1, \\ \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1}(u_0), \mathsf{p}(\tau_1)\big) & \text{for} \quad \tau_1 \le t < \tau_2. \end{cases}$$

Thus, it follows that for $t \in [0,\tau_1]$,

$$\begin{aligned} u_{0,t}(u_0) =& u_0 + \sum_{i=1}^{k} \int_0^{t} A_s^i(\tilde{u}_{0,s}(u_0))ds + \sum_{j=1}^{\infty} \int_0^{t} B_s^j(\tilde{u}_{0,s}(u_0))dW_s^j \\ &+ \int_0^{t} \int_{\mathcal{D}^c} \gamma_s(\tilde{u}_{0,s}(u_0), z)\tilde{N}(ds,dz) + \int_0^{t} \int_{\mathcal{D}} \gamma_s(\tilde{u}_{0,s}(u_0), z)N(ds,dz) \end{aligned}$$

implying that the process $u_{0,t}(u_0)$ is an H-valued solution to SEE (3.3) on $[0,\tau_1]$. Since $\gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1}(u_0), \mathsf{p}(\tau_1)\big) = \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1^-}(u_0), \mathsf{p}(\tau_1)\big)$, the uniqueness of the solution $u_{0,t}(u_0)$ on $[0,\tau_1]$ follows from the uniqueness of the solution $\tilde{u}_{0,t}(u_0)$ on $[0,\tau_1)$.

Further, let $\tilde{u}_{\tau_1,t}\big(u_{0,\tau_1}(u_0)\big)$ be the unique H-valued solution to SEE (3.7) in the interval $[\tau_1,T]$ with initial condition $u_{0,\tau_1}(u_0)$ obtained using Lemma 3.5. Thus,

$$\begin{aligned} \tilde{u}_{\tau_1,t}\big(u_{0,\tau_1}(u_0)\big) =& u_{0,\tau_1}(u_0) + \sum_{i=1}^{k} \int_{\tau_1}^{t} A_s^i\big(\tilde{u}_{\tau_1,s}\big(u_{0,\tau_1}(u_0)\big)\big)ds + \sum_{j=1}^{\infty} \int_{\tau_1}^{t} B_s^j\big(\tilde{u}_{\tau_1,s}\big(u_{0,\tau_1}(u_0)\big)\big)dW_s^j \\ &+ \int_{\tau_1}^{t} \int_{\mathcal{D}^c} \gamma_s\big(\tilde{u}_{\tau_1,s}\big(u_{0,\tau_1}(u_0)\big), z\big)\tilde{N}(ds,dz) \end{aligned}$$

for $t \in [\tau_1,T]$. We construct a solution to SEE (3.3) on $[0,\tau_2]$ as follows:

$$u_{0,t}(u_0) = \begin{cases} u_{0,t}(u_0) & \text{for} \quad 0 \le t \le \tau_1, \\ \tilde{u}_{\tau_1,t}\big(u_{0,\tau_1}(u_0)\big) & \text{for} \quad \tau_1 \le t < \tau_2, \\ \tilde{u}_{\tau_1,\tau_2^-}\big(u_{0,\tau_1}(u_0)\big) + \gamma_{\tau_2^-}\big(\tilde{u}_{\tau_1,\tau_2^-}\big(u_{0,\tau_1}(u_0)\big), \mathsf{p}(\tau_2)\big) & \text{for} \quad t = \tau_2. \end{cases}$$

Using the similar argument as above and observing that

$$\begin{aligned} \int_0^{\tau_2} \int_{\mathcal{D}} \gamma_s(\tilde{u}_{0,s}(u_0), z)N(ds,dz) &= \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1}(u_0), \mathsf{p}(\tau_1)\big) + \gamma_{\tau_2}\big(\tilde{u}_{0,\tau_2}(u_0), \mathsf{p}(\tau_2)\big) \\ &= \gamma_{\tau_1}\big(\tilde{u}_{0,\tau_1^-}(u_0), \mathsf{p}(\tau_1)\big) + \gamma_{\tau_2}\big(\tilde{u}_{\tau_1,\tau_2^-}\big(u_{0,\tau_1}(u_0)\big), \mathsf{p}(\tau_2)\big), \end{aligned}$$

we obtain that $u_{0,t}(u_0)$ is a unique solution of SEE (3.3) on the interval $[0,\tau_2]$, where the uniqueness of the solution follows from the uniqueness of the solutions $\tilde{u}_{0,t}(u_0)$ and $\tilde{u}_{\tau_1,t}\big(u_{0,\tau_1}(u_0)\big)$ of SEE (3.7) on the intervals $[0,T]$ and $[\tau_1,T]$ respectively.

Continuing this interlacing procedure successively, a unique solution to SEE (3.3) can be constructed on the interval $[0,\tau_n]$ for every $n \in \mathbb{N}$ and hence on $[0,T]$.

Chapter 4

Semilinear SPDEs with monotone semilinear term

In this chapter, we consider semilinear stochastic partial differential equations on bounded domains $\mathscr{D}$. The semilinear term may have arbitrary polynomial growth as long as it is continuous and monotone except perhaps near the origin. Typical examples are the stochastic Allen–Cahn and Ginzburg–Landau equations. The first main result of this chapter are L^p-estimates for such equations. The L^p-estimates are subsequently employed in obtaining higher regularity. This is motivated by ongoing work to obtain rate of convergence estimates for numerical approximations to such equations. It is shown, under appropriate assumptions, that the solution is continuous in time with values in the Sobolev space $W^{2,2}(\mathscr{D}')$ and L^2-integrable with values in $W^{3,2}(\mathscr{D}')$, for any compact $\mathscr{D}' \subset \mathscr{D}$. Using results from L^p-theory of SPDEs obtained by Kim [20], we get analogous results in weighted Sobolev spaces on the whole $\mathscr{D}$. Finally it is shown that the solution is Hölder continuous in time of order $\gamma < \frac{1}{2} - \frac{2}{q}$ as a process with values in a weighted L^q-space, where q arises from the integrability assumptions imposed on the initial condition and forcing terms. The work presented in this chapter is based on my joint work [37].

The chapter is organised as follows: Section 4.1 is devoted to the proof of Theorem 4.1 which gives us the desired L^p-estimates for the solution to semilinear SPDE (4.1). In Section 4.2, we first prove interior regularity for the associated linear SPDE, see Theorem 4.3. We then use the results on interior regularity for the linear SPDE to prove interior regularity for the semilinear SPDE (4.1) in Theorem 4.2. Remark 4.5 explains why we can not prove higher regularity than that given by Theorem 4.2. In Section 4.3, we prove regularity results up to the boundary and time regularity in weighted Sobolev spaces using L^p-theory from Kim [20], see Theorems 4.5 and 4.6. Finally in Section 4.4, we have shown with the help of an example that raising the regularity twice is enough to get the rate of convergence estimates for the proposed numerical scheme. The main results and required assumptions are stated at the beginning of each section.

4.1 L^p-estimates for the semilinear equation

Let $T > 0$ be given, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis, $\mathscr{P}$ be the predictable σ-algebra and $W := (W_t)_{t\in[0,T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. Further, let $\mathscr{D}$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Besides the notations introduced in Section 1.2, we introduce two more notations. If $h \in L^p((0,T); L^p(\mathscr{D}))$, then we use $\|h\|_{L^p}$ to denote the norm and $\mathscr{D}' \Subset \mathscr{D}$ signifies that $\mathscr{D}'$ is a compact subset of $\mathscr{D}$.

We consider the following semilinear SPDE :

$$
\begin{aligned}
&du_t = (L_t u_t + f_t(u_t, \nabla u_t) + f_t^0)dt + \sum_{k\in\mathbb{N}} (M_t^k u_t + g_t^k) dW_t^k \quad \text{on } [0,T] \times \mathscr{D}, \\
&u_t = 0 \text{ on } \partial\mathscr{D}, \quad u_0 = \phi \text{ on } \mathscr{D},
\end{aligned}
\tag{4.1}
$$

where,

$$L_t u := \sum_{j=1}^{d} \partial_j \Big(\sum_{i=1}^{d} a_t^{ij} \partial_i u \Big) + \sum_{i=1}^{d} b_t^i \partial_i u + c_t u \quad \text{and} \quad M_t^k u := \sum_{i=1}^{d} \sigma_t^{ik} \partial_i u + \mu_t^k u. \tag{4.2}$$

Fix constants $K > 0$, $\kappa > 0$ and $\alpha \geq 2$. We assume the following:

A - 4.1. For any $i, j = 1, \dots, d$, the coefficients a^{ij}, b^i and c are real-valued, $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and are bounded by K. The coefficients $\sigma^i = (\sigma^{ik})_{k=1}^{\infty}$, $\mu = (\mu^k)_{k=1}^{\infty}$ are ℓ^2-valued, $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and almost surely,

$$\sum_{i=1}^{d} \sum_{k \in \mathbb{N}} |\sigma_t^{ik}(x)|^2 + \sum_{k \in \mathbb{N}} |\mu_t^k(x)|^2 \leq K \quad \forall t \in [0,T],\ x \in \mathscr{D}.$$

A - 4.2. Almost surely,

$$\sum_{i,j=1}^{d} \Big(a_t^{ij}(x) - \frac{1}{2} \sum_{k \in \mathbb{N}} \sigma_t^{ik}(x) \sigma_t^{jk}(x) \Big) \xi_i \xi_j \geq \kappa |\xi|^2 \quad \forall t \in [0,T],\ x \in \mathscr{D},\ \xi \in \mathbb{R}^d.$$

A - 4.3. The function $f = f_t(\omega, x, r, z)$ is $\mathscr{P} \times \mathscr{B}(\mathscr{D}) \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)$-measurable, it is continuous in (r, z) almost surely for all t and x. Furthermore, almost surely

$$\begin{aligned} (r - r')(f_t(x,r,z) - f_t(x,r',z)) &\leq K|r - r'|^2, \\ |f_t(x,r,z) - f_t(x,r,z')| &\leq K|z - z'|, \\ |f_t(x,r,z)| &\leq K(1 + |r|)^{\alpha - 1} \end{aligned}$$

for all $t \in [0,T],\ x \in \mathscr{D},\ r, r' \in \mathbb{R},\ z, z' \in \mathbb{R}^d$.

A - 4.4. For some $p \geq \alpha$,

$$\phi \in L^p(\Omega; L^p(\mathscr{D})),\ \ f^0 \in L^p(\Omega \times (0,T); L^p(\mathscr{D})) \text{ and } g \in L^p(\Omega \times (0,T); L^p(\mathscr{D}; \ell^2)).$$

Remark 4.1. Without loss of generality, we may assume that almost surely for all t, x and z the function $r \mapsto f_t(x,r,z)$ is decreasing. If not, then (4.1) can be rewritten by replacing $f_t(x,r,z)$ with $\bar{f}_t(x,r,z) := f_t(x,r,z) - Kr$ and $c_t(x)$ with $\bar{c}_t(x) := c_t(x) + K$, where using Assumption A-4.3,

$$(r - r')(\bar{f}_t(x,r,z) - \bar{f}_t(x,r',z)) = (r - r')(f_t(x,r,z) - f_t(x,r',z)) - K|r - r'|^2 \leq 0$$

showing that $\bar{f}$ is decreasing in r. Further, we may assume that almost surely for all t and x, $f_t(x,0,0) = 0$. Otherwise, we can replace $f_t(x,r,z)$ in (4.1) by $\tilde{f}_t(x,r,z) := f_t(x,r,z) - f_t(x,0,0)$ and f_t^0 by $\tilde{f}_t^0(x) := f_t^0(x) + f_t(x,0,0)$.

Definition 4.1 (L^2-Solution)**.** An adapted, continuous $L^2(\mathscr{D})$-valued process is said to be a solution of SPDE (4.1) if

(i) $dt \times \mathbb{P}$ almost everywhere $u \in L^\alpha(\mathscr{D}) \cap W_0^{1,2}(\mathscr{D})$ and

$$\mathbb{E} \int_0^T (|u_t|_{L^\alpha}^\alpha + |u_t|_{W_0^{1,2}}^2)\, dt < \infty,$$

(ii) almost surely for every $t \in [0,T]$ and $\xi \in C_0^\infty(\mathscr{D})$,

$$(u_t, \xi) = (u_0, \xi) + \int_0^t \langle L_s(u_s) + f_s(u_s, \nabla u_s) + f_s^0, \xi \rangle ds + \sum_{k \in \mathbb{N}} \int_0^t (\xi, M_s^k(u_s) + g_s^k) dW_s^k.$$

The following theorem is the main result of this section.

Theorem 4.1. *If Assumptions A-4.1 to A-4.4 hold, then there exists a unique solution u to (4.1) and*

$$\mathbb{E}\sup_{0\le t\le T}|u_t|^p_{L^p}+\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u_s|^2|u_s|^{p-2}dxds\le C\mathbb{E}\Big(|\phi|^p_{L^p}+\|f^0\|^p_{L^p}+\||g|_{\ell^2}\|^p_{L^p}\Big),\tag{4.3}$$

where $C=C(d,p,K,\kappa,T)$.

The rest of Section 4.1 is devoted to proving Theorem 4.1 but we give a brief outline of the proof here.

1. We replace the semilinear term f by truncations f^m, depending on some $m\in\mathbb{N}$, chosen in such a way that the monotonicity is preserved and f^m are bounded. From standard theory of stochastic evolution equations we obtain u^m which are solutions to the SPDE with f replaced with f^m.

2. We now wish to get the estimate (4.3) for these u^m (uniformly in m). If we were allowed to apply Itô's formula directly to $r\mapsto|r|^p$ and the process $u^m_t(x)$ and to integrate over $\mathscr{D}$ then (4.3) for u^m would follow from A-4.1, A-4.2 and A-4.3.

3. Since, of course, this is not allowed we instead consider an appropriate bounded smooth approximation ϕ_n to $r\mapsto|r|^p$ and use the Itô's formula from Krylov [27]. We then establish an estimate similar to (4.3) but for $\phi_n(u^m)$ instead of $|u^m|^p$ and with the right-hand-side still depending on m but independent of n. See Lemma 4.2. This allows us to take the limit $n\to\infty$ and to use the monotonicity of $r\mapsto f^m_t(x,r,z)$ to obtain (4.3) for u^m. See Lemma 4.3.

4. The final step is then to use compactness argument to obtain u as a weak limit of $(u^m)_{m\in\mathbb{N}}$, see Lemma 4.4, and the usual monotonicity argument to show that u satisfies (4.1). Fatou's lemma will then yield (4.3) for u.

Before proceeding with the proof of Theorem 4.1, we observe the following:

Remark 4.2. Assumptions A-4.1 and A-4.2 imply, the existence of a constant K' depending on K,d and κ only such that almost surely for all $t\in[0,T]$ and $w,w'\in W^{1,2}_0(\mathscr{D})$,

$$2\langle L_tw+f^0_t,w\rangle+\sum_{k\in\mathbb{N}}|M^k_tw+g^k_t|^2_{L^2}+\kappa|w|^2_{W^{1,2}_0}\le K'\Big[|f^0_t|^2_{L^2}+\big||g_t|_{\ell^2}\big|^2_{L^2}+|w|^2_{L^2}\Big]$$

and

$$2\langle L_tw-L_tw',w-w'\rangle+\sum_{k\in\mathbb{N}}|M^k_tw-M^k_tw'|^2_{L^2}+\kappa|w-w'|^2_{W^{1,2}_0}\le K'|w-w'|^2_{L^2}\ .$$

Indeed, substituting the values of the operators L and M and then using integration by parts, we have

$$\begin{aligned}&2\langle L_tw+f^0_t,w\rangle+\sum_{k\in\mathbb{N}}|M^k_tw+g^k_t|^2_{L^2}\\
=&2\int_{\mathscr{D}}\sum_{j=1}^d\partial_j\Big(\sum_{i=1}^d a^{ij}_t(x)\partial_iw(x)\Big)w(x)dx+2\int_{\mathscr{D}}\sum_{i=1}^d\big(b^i_t(x)\partial_iw(x)+c_t(x)w(x)+f^0_t(x)\big)w(x)dx\\
&+\sum_{k\in\mathbb{N}}\int_{\mathscr{D}}\Big|\sum_{i=1}^d\big(\sigma^{ik}_t(x)\partial_iw(x)+\mu^k_t(x)w(x)+g^k_t(x)\big)\Big|^2dx\\
\le&-2\int_{\mathscr{D}}\sum_{i,j=1}^d a^{ij}_t(x)\partial_iw(x)\partial_jw(x)dx+2\int_{\mathscr{D}}\sum_{i=1}^d\big(b^i_t(x)\partial_iw(x)+c_t(x)w(x)+f^0_t(x)\big)w(x)dx\end{aligned}$$

$$+\sum_{k\in\mathbb{N}}\int_{\mathscr{D}}\sum_{i,j=1}^{d}\sigma_t^{ik}(x)\sigma_t^{jk}(x)\partial_i w(x)\partial_j w(x)dx+2\sum_{k\in\mathbb{N}}\int_{\mathscr{D}}\sum_{i=1}^{d}\sigma_t^{ik}(x)\partial_i w(x)\mu_t^k(x)w(x)dx$$

$$+2\sum_{k\in\mathbb{N}}\int_{\mathscr{D}}\sum_{i=1}^{d}\sigma_t^{ik}(x)\partial_i w(x)g_t^k(x)dx+2\int_{\mathscr{D}}\sum_{k\in\mathbb{N}}|\mu_t^k(x)|^2|w(x)|^2dx+2\int_{\mathscr{D}}\sum_{k\in\mathbb{N}}|g_t^k(x)|^2dx$$

which on using Assumptions A-4.1, A-4.2 and Young's inequality is further

$$\begin{aligned}\le &-2\kappa\int_{\mathscr{D}}|\nabla w(x)|^2dx+\epsilon\int_{\mathscr{D}}\Big(\sum_{i=1}^{d}|\partial_i w(x)|^2+|w(x)|^2+|f_t^0(x)|^2\Big)dx\\&+\epsilon\sum_{k\in\mathbb{N}}\int_{\mathscr{D}}\Big|\sum_{i=1}^{d}\sigma_t^{ik}(x)\partial_i w(x)\Big|^2dx+C\int_{\mathscr{D}}|w(x)|^2dx+C\big||g_t|_{\ell^2}\big|_{L^2}^2\,.\end{aligned}$$

Note that using Assumption A-4.1, we have

$$\sum_{k\in\mathbb{N}}\Big|\sum_{i=1}^{d}\sigma_t^{ik}(x)\partial_i w(x)\Big|^2\le K\sum_{i=1}^{d}\Big(\sum_{k\in\mathbb{N}}|\sigma_t^{ik}(x)|^2\Big)|\partial_i w(x)|^2\le K|\nabla w(x)|^2$$

and therefore,

$$2\langle L_t w+f_t^0,w\rangle+\sum_{k\in\mathbb{N}}|M_t^k w+g_t^k|_{L^2}^2\le-\kappa|\nabla w|_{L^2}^2+C\big[|f_t^0|_{L^2}^2+\big||g_t|_{\ell^2}\big|_{L^2}^2+|w|_{L^2}^2\big]$$

proving the first part of the statement.

Second part follows by repeating the above calculations replacing w by $w-w'$, f_t^0 and g_t^k by 0 and then using the linearity of the operators L and M_t^k, $k\in\mathbb{N}$.

In the following lemma we show that the solution to (4.1), if exists, is unique.

Lemma 4.1 (Uniqueness)**.** *The solution to* (4.1) *is unique in the sense that if u and $\bar{u}$ both satisfy* (4.1) *then*

$$\mathbb{P}\Big(\sup_{t\le T}|u_t-\bar{u}_t|_{L^2}=0\Big)=1.$$

Proof. Let u and $\bar{u}$ be two solutions of (4.1) in the sense of Definition 4.1. Then,

$$\begin{aligned}u_t-\bar{u}_t=&\int_0^t\left(L_s(u_s)-L_s(\bar{u}_s)+f_s(u_s,\nabla u_s)-f_s(\bar{u}_s,\nabla\bar{u}_s)\right)ds\\&+\sum_{k\in\mathbb{N}}\int_0^t\left(M_s^k(u_s)-M_s^k(\bar{u}_s)\right)dW_s^k\end{aligned}\tag{4.4}$$

almost surely for all $t\in[0,T]$. Using Remark 4.1, Assumption A-4.3 and Young's inequality, we get

$$\begin{aligned}&\langle f_t(u_t,\nabla u_t)-f_t(\bar{u}_t,\nabla\bar{u}_t),u_t-\bar{u}_t\rangle\\&=\langle f_t(u_t,\nabla u_t)-f_t(\bar{u}_t,\nabla u_t)+f_t(\bar{u}_t,\nabla u_t)-f_t(\bar{u}_t,\nabla\bar{u}_t),u_t-\bar{u}_t\rangle\\&\le\frac{\kappa}{2}|\nabla(u_t-\bar{u}_t)|_{L^2}^2+C|u_t-\bar{u}_t|_{L^2}^2\,,\end{aligned}\tag{4.5}$$

for some constant C. Let $K''=K'+C$, where K' is the constant obtained in Remark 4.2. Then using the product rule and applying Itô's formula for the the square of the norm to (4.4), see Gyöngy and Šiška [16, Theorem 2.1] or Pardoux [39, Chapitre 2, Theoreme 5.2], we obtain

$$\begin{aligned}d\Big(e^{-K''t}|u_t-\bar{u}_t|_{L^2}^2\Big)&=e^{-K''t}\big[d|u_t-\bar{u}_t|_{L^2}^2-K''|u_t-\bar{u}_t|_{L^2}^2\,dt\big]\\&=e^{-K''t}\Big[\Big(2\langle L_t(u_t)-L_t(\bar{u}_t)+f_t(u_t,\nabla u_t)-f_t(\bar{u}_t,\nabla\bar{u}_t),u_t-\bar{u}_t\rangle\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\in\mathbb{N}}|M^k_t(u_t)-M^k_t(\bar u_t)|^2_{L^2}-K''|u_t-\bar u_t|^2_{L^2}\Big)\,dt \\
&+\sum_{k\in\mathbb{N}}2\big(u_t-\bar u_t,M^k_t(u_t)-M^k_t(\bar u_t)\big)dW^k_t\Big]
\end{aligned}
\tag{4.6}
$$

almost surely for all $t\in[0,T]$. Substituting (4.5) in (4.6) and using Remark 4.2, we get

$$
e^{-K''t}|u_t-\bar u_t|^2_{L^2}\le 2\sum_{k\in\mathbb{N}}\int_0^t e^{-K''s}\big(u_s-\bar u_s,M^k_s(u_s)-M^k_s(\bar u_s)\big)dW^k_s
$$

implying that right hand side is a non-negative local martingale (and thus a super-martingale) starting from 0 and hence for all $t\in[0,T]$,

$$
\mathbb{E}[e^{-K''t}|u_t-\bar u_t|^2_{L^2}]\le 0.
$$

Thus for all $t\in[0,T]$, we get $\mathbb{P}(|u_t-\bar u_t|^2_{L^2}=0)=1$ which, along with the continuity of $u-\bar u$ in $L^2(\mathscr{D})$, concludes the proof. □

Having proved uniqueness we start preparing the proof of Theorem 4.1. For $m\in\mathbb{N}$, consider the truncated function

$$
f^m_t(x,r,z)=\begin{cases} f_t(x,-m,z) & \text{if } r<-m\\ f_t(x,r,z) & \text{if } -m\le r\le m\\ f_t(x,m,z) & \text{if } r>m,\end{cases}
$$

and the equation

$$
\begin{aligned}
&du^m_t=(L_tu^m_t+f^m_t(u^m_t,\nabla u^m_t)+f^0_t)dt+\sum_{k\in\mathbb{N}}(M^k_tu^m_t+g^k_t)dW^k_t,\\
&u^m_t=0 \text{ on } \partial\mathscr{D},\quad u^m_0=\phi \text{ on } \mathscr{D}.
\end{aligned}
\tag{4.7}
$$

For each $m\in\mathbb{N}$, using Assumption A-4.3, $f^m_t(x,r,z)$ is bounded and hence (4.7) can be viewed as a SPDE on the Gelfand triple $W^{1,2}_0(\mathscr{D})\hookrightarrow L^2(\mathscr{D})\hookrightarrow W^{-1,2}(\mathscr{D})$ and all the conditions for existence and uniqueness of solution in [28] are satisfied. Thus (4.7) has a unique L^2-solution in the sense of [28, Chapter 2, Definition 2.2].

We now prove an estimate similar to (4.3) for the solutions of (4.7). We will do this by applying the Itô's formula from Krylov [27] similarly to Dareiotis and Gerencsér [6]. To that end we need to consider the functions,

$$
\phi_n(r)=\begin{cases} |r|^p & \text{if } |r|<n\\ n^{p-2}\frac{p(p-1)}{2}(|r|-n)^2+pn^{p-1}(|r|-n)+n^p & \text{if } |r|\ge n.\end{cases}
$$

Note that,

$$
\phi'_n(r)=\begin{cases} p|r|^{p-2}r & \text{if } |r|<n\\ n^{p-2}p(p-1)(r-n)+pn^{p-1} & \text{if } r\ge n\\ n^{p-2}p(p-1)(r+n)-pn^{p-1} & \text{if } r\le -n\end{cases}
$$

and,

$$
\phi''_n(r)=\begin{cases} p(p-1)|r|^{p-2} & \text{if } |r|<n\\ n^{p-2}p(p-1) & \text{if } r\ge n\\ n^{p-2}p(p-1) & \text{if } r\le -n\end{cases}
$$

Thus, we see that ϕ_n are twice continuously differentiable and for any $r\in\mathbb{R}$,

$$
|\phi_n(r)|\le C|r|^2,\ |\phi'_n(r)|\le C|r|,\ |\phi''_n(r)|\le C
$$

where C depends on p and $n\in\mathbb{N}$ only. Further,

$$
\phi_n(r)\to|r|^p,\ \phi'_n(r)\to p|r|^{p-2}r,\ \phi''_n(r)\to p(p-1)|r|^{p-2}
\tag{4.8}
$$

as $n \to \infty$ and

$$\phi_n(r) \le C|r|^p,\ \phi_n'(r) \le C|r|^{p-1},\ \phi_n''(r) \le C|r|^{p-2}, \tag{4.9}$$

where C depends on p only.

Remark 4.3. With some simple calculations, we obtain for any $r \in \mathbb{R}$,

(a) $|r\phi_n'(r)| \le p\phi_n(r)$,

(b) $|r^2\phi_n''(r)| \le p(p-1)\phi_n(r)$,

(c) $|\phi_n'(r)|^2 \le 4p\phi_n''(r)\phi_n(r)$,

(d) $|\phi_n''(r)|^{\frac{p}{p-2}} \le [p(p-1)]^{\frac{p}{p-2}}\phi_n(r)$.

Further, using these inequalities we get

(i) $|u_s^m\phi_n'(u_s^m)| \le C\phi_n(u_s^m)$,

(ii) $|u_s^m|^2\phi_n''(u_s^m) \le C\phi_n(u_s^m)$,

(iii) $\sum_{i=1}^d \partial_i u_s^m\phi_n'(u_s^m) \le \epsilon\phi_n''(u_s^m)|\nabla u_s^m|^2 + C\phi_n(u_s^m)$,

(iv) $|f_s^0\phi_n'(u_s^m)| \le C|f_s^0|^p + C\phi_n(u_s^m)$,

(v) $|f_s^m(u_s^m, \nabla u_s^m)\phi_n'(u_s^m)| \le C|f_s(-m, \nabla u_s^m)|^p + C\phi_n(u_s^m)$,

(vi) $|g_s|_{\ell^2}^2\phi_n''(u_s^m) \le C\phi_n(u_s^m) + C|g_s|_{\ell^2}^p$.

where C depends only on d, p and ϵ.

Indeed, (i) follows from (a), (ii) follows from (b). Further, using (c) and then applying Young's inequality, we get

$$\partial_i u_s^m\phi_n'(u_s^m) \le C|\partial_i u_s^m|[\phi_n''(u_s^m)]^{\frac{1}{2}}[\phi_n(u_s^m)]^{\frac{1}{2}} \le \epsilon|\partial_i u_s^m|^2\phi_n''(u_s^m) + C\phi_n'(u_s^m)$$

which on taking summation over i yields (iii). Again, using (c) and applying Young's inequality twice, we get

$$\begin{aligned}
|f_s^0\phi_n'(u_s^m)| &\le C|f_s^0|[\phi_n''(u_s^m)]^{\frac{1}{2}}[\phi_n(u_s^m)]^{\frac{1}{2}} \\
&\le C|f_s^0|^2\phi_n''(u_s^m) + C\phi_n(u_s^m) \\
&\le C|f_s^0|^p + C[\phi_n''(u_s^m)]^{\frac{p}{p-2}} + C\phi_n(u_s^m)
\end{aligned}$$

which on using (d) gives (iv). Similarly,

$$\begin{aligned}
|f_s^m(u_s^m, \nabla u_s^m)\phi_n'(u_s^m)| &\le C|f_s^m(u_s^m, \nabla u_s^m)|[\phi_n''(u_s^m)]^{\frac{1}{2}}[\phi_n(u_s^m)]^{\frac{1}{2}} \\
&\le C|f_s^m(u_s^m, \nabla u_s^m)|^p + C\phi_n(u_s^m) \\
&\le C|f_s(-m, \nabla u_s^m)|^p + C\phi_n(u_s^m)
\end{aligned}$$

where the last inequality follows from the monotonicity of function f^m. Finally, using Young's inequality and then using (d), we obtain

$$|g_s|_{\ell^2}^2\phi_n''(u_s^m) \le C|g_s|_{\ell^2}^p + C[\phi_n''(u_s^m)]^{\frac{p}{p-2}} \le C\phi_n(u_s^m) + C|g_s|_{\ell^2}^p$$

as desired.

Remark 4.3 will be very useful in what follows. Using Theorem 3.1 from [27], we get that almost surely

$$\int_{\mathscr{D}} \phi_n(u_t^m)dx = \int_{\mathscr{D}} \phi_n(u_0^m)dx + \sum_{k\in\mathbb{N}}\int_0^t\int_{\mathscr{D}}\Big(\sum_{i=1}^d \sigma_s^{ik}\partial_i u_s^m + \mu_s^k u_s^m + g_s^k\Big)\phi_n'(u_s^m)dxdW_s^k$$

$$+\int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d b_s^i \partial_i u_s^m + c_s u_s^m + f_s^m(u_s^m, \nabla u_s^m) + f_s^0 \Big) \phi_n'(u_s^m) dx ds$$
$$- \int_0^t \int_{\mathscr{D}} \sum_{i,j=1}^d a_s^{ij} \partial_i u_s^m \phi_n''(u_s^m) \partial_j u_s^m dx ds$$
$$+ \frac{1}{2} \int_0^t \int_{\mathscr{D}} \sum_{k\in\mathbb{N}} \Big| \sum_{i=1}^d \sigma_s^{ik} \partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big|^2 \phi_n''(u_s^m) dx ds,$$

for any $t \in [0, T]$ and $n \in \mathbb{N}$. Thus using Assumptions A-4.1, A-4.2 and Young's inequality for any $\epsilon > 0$, we obtain almost surely

$$\begin{aligned}
&\int_{\mathscr{D}} \phi_n(u_t^m) dx \leq \int_{\mathscr{D}} \phi_n(u_0^m) dx + \mathscr{M}_t^{n,m} \\
&+ \int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d b_s^i \partial_i u_s^m + c_s u_s^m + f_s^m(u_s^m, \nabla u_s^m) + f_s^0 \Big) \phi_n'(u_s^m) dx ds \\
&- \int_0^t \int_{\mathscr{D}} \kappa |\nabla u_s^m|^2 \phi_n''(u_s^m) dx ds + \int_0^t \int_{\mathscr{D}} \Big(\epsilon |\nabla u_s^m|^2 + C|u_s|^2 + C|g_s|_{\ell^2}^2 \Big) \phi_n''(u_s^m)\, dx ds,
\end{aligned} \tag{4.10}$$

for any $t \in [0, T]$ and $n \in \mathbb{N}$. Here the generic constant C depends only on d, K and ϵ and

$$\mathscr{M}_t^{n,m} := \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d \sigma_s^{ik} \partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big) \phi_n'(u_s^m) dx dW_s^k$$

is a local martingale. Further, using Burkholder–Davis–Gundy's inequality, Remark 4.3(c) and Hölder's inequality, we see that

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq t\leq T} |\mathscr{M}_t^{n,m}| \\
&\leq C\mathbb{E} \bigg(\int_0^T \sum_k \Big(\int_{\mathscr{D}} \Big| \sum_{i=1}^d \sigma_s^{ik} \partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big| \Big(\phi_n''(u_s^m) \phi_n(u_s^m) \Big)^{\frac{1}{2}} dx \Big)^2 ds \bigg)^{\frac{1}{2}} \\
&\leq C\mathbb{E} \bigg(\int_0^T \Big(\sum_k \int_{\mathscr{D}} \Big| \sum_{i=1}^d \sigma_s^{ik} \partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big|^2 \phi_n''(u_s^m) dx \int_{\mathscr{D}} \phi_n(u_s^m) dx \Big) ds \bigg)^{\frac{1}{2}}
\end{aligned}$$

which, using the same steps as before, in particular Remark 4.3 points (ii) and (vi), gives

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq t\leq T} |\mathscr{M}_t^{n,m}| \\
&\leq C\mathbb{E} \bigg(\int_0^T \Big(\int_{\mathscr{D}} \Big(|\nabla u_s^m|^2 + |u_s^m|^2 + |g_s|_{\ell^2}^2 \Big) \phi_n''(u_s^m) dx \int_{\mathscr{D}} \phi_n(u_s^m) dx \Big) ds \bigg)^{\frac{1}{2}} \\
&\leq C\mathbb{E} \bigg(\sup_{0\leq t\leq T} \int_{\mathscr{D}} \phi_n(u_t^m) dx \int_0^T \int_{\mathscr{D}} \Big[|\nabla u_s^m|^2 \phi_n''(u_s^m) + \phi_n(u_s^m) + |g_s|_{\ell^2}^p \Big] dx ds \bigg)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \mathbb{E} \sup_{0\leq t\leq T} \int_{\mathscr{D}} \phi_n(u_t^m) dx + C\mathbb{E} \int_0^T \int_{\mathscr{D}} \Big[|\nabla u_s^m|^2 \phi_n''(u_s^m) + \phi_n(u_s^m) + |g_s|_{\ell^2}^p \Big] dx ds
\end{aligned} \tag{4.11}$$

Lemma 4.2. *If u^m is the solution to* (4.7), *then*

$$\begin{aligned}
\mathbb{E} \sup_{0\leq t\leq T} |u_t^m|_{L^p}^p + \mathbb{E} \int_0^t \int_{\mathscr{D}} |\nabla u_s^m|^2 |u_s^m|^{p-2} dx ds \\
\leq C\mathbb{E} \Big(|\phi|_{L^p}^p + C_m + \|f^0\|_{L^p}^p + \||g|_{\ell^2}\|_{L^p}^p \Big),
\end{aligned} \tag{4.12}$$

where $C = C(d, K, \kappa, p, T)$ *and* $C_m := \mathbb{E}\int_0^T \int_{\mathscr{D}} (1+|m|)^{(\alpha-1)p} dxds$ *are constants.*

Proof. Taking expectation in (4.10) and using Remark 4.3 along with Assumptions A-4.1, A-4.3 and (4.9), we get

$$\begin{aligned}
\mathbb{E}\int_{\mathscr{D}} \phi_n(u_t^m)dx + \frac{\kappa}{2}\mathbb{E}\int_0^t \int_{\mathscr{D}} |\nabla u_s^m|^2 \phi_n''(u_s^m)dxds \leq C\mathbb{E}\int_{\mathscr{D}} \phi_n(u_0^m)dx + C_m \\
+ \mathbb{E}\int_0^t \int_{\mathscr{D}} |f_s^0|^p dxds + C\mathbb{E}\int_0^t \int_{\mathscr{D}} |g_s|_{\ell^2}^p dxds + C\int_0^t \mathbb{E}\int_{\mathscr{D}} \phi_n(u_s^m)dxds \\
\leq C\mathbb{E}\mathcal{K}_t^m + C\int_0^t \mathbb{E}\int_{\mathscr{D}} \phi_n(u_s^m)dxds,
\end{aligned}$$

where $C = C(d, p, K, \kappa)$ and

$$\mathcal{K}_t^m := \int_{\mathscr{D}} |\phi|^p dx + C_m + \int_0^t \int_{\mathscr{D}} |f_s^0|^p dxds + \int_0^t \int_{\mathscr{D}} |g_s|_{\ell^2}^p \, dxds.$$

Applying Gronwall's lemma, we obtain for any $t \in [0, T]$

$$\mathbb{E}\int_{\mathscr{D}} \phi_n(u_t^m)dx + \mathbb{E}\int_0^t \int_{\mathscr{D}} |\nabla u_s^m|^2 \phi_n''(u_s^m)dxds \leq C\mathbb{E}\mathcal{K}_t^m$$

where $C = C(d, p, K, \kappa, T)$.

Further, taking the supremum over $t \in [0, T]$ in (4.10), using the same estimates as given above and then taking expectation, we get using (4.11)

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \phi_n(u_t^m)dx \\
\leq &C\mathbb{E}\int_{\mathscr{D}} \phi_n(u_0^m)dx + \mathbb{E}\sup_{0\leq t\leq T}\int_0^t \int_{\mathscr{D}} f_s^m(u_s^m, \nabla u_s^m)\phi_n'(u_s^m)dxds \\
&+ C\mathbb{E}\int_0^T \int_{\mathscr{D}} |f_s^0|^p dxds + C\mathbb{E}\int_0^T \int_{\mathscr{D}} |g_s|_{\ell^2}^p \, dxds + C\int_0^T \mathbb{E}\int_{\mathscr{D}} \phi_n(u_s^m)dxds \\
&+ \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \phi_n(u_t^m)dx + C\mathbb{E}\int_0^T \int_{\mathscr{D}} \Big[|\nabla u_s^m|^2 \phi_n''(u_s^m) + \phi_n(u_s^m)\Big]dxds \\
\leq &C\mathbb{E}\int_{\mathscr{D}} \phi_n(u_0^m)dx + CC_m + C\mathbb{E}\int_0^T \int_{\mathscr{D}} |f_s^0|^p dxds + C\mathbb{E}\int_0^T \int_{\mathscr{D}} \big[|g_s|_{\ell^2}^p + \phi_n(u_s^m)\big] \, dxds \\
&+ \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \phi_n(u_t^m)dx + C\mathbb{E}\int_0^T \int_{\mathscr{D}} |\nabla u_s^m|^2 \phi_n''(u_s^m)dxds \\
\leq &C\mathbb{E}\mathcal{K}_T^m + \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \phi_n(u_t^m)dx < \infty
\end{aligned}$$

where C does not depend on n and m. Thus, we have

$$\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \phi_n(u_t^m)dx + \mathbb{E}\int_0^T \int_{\mathscr{D}} |\nabla u_s^m|^2 \phi_n''(u_s^m)dxds \leq C\mathbb{E}\mathcal{K}_T^m < \infty,$$

where $C = C(d, p, K, \kappa, T)$. Now we let $n \to \infty$ and apply Fatou's lemma to complete the proof. □

We can now use Lemma 4.2 and the monotonicity of $r \mapsto f_t^m(x, r, z)$ to obtain an estimate for u_t^m, where the right-hand-side no longer depends on m. Let

$$\mathscr{K}_t := \int_{\mathscr{D}} |\phi|^p dx + \int_0^t \int_{\mathscr{D}} \big[|f_s^0|^p + |g_s|_{\ell^2}^p\big] \, dxds.$$

Lemma 4.3. *If u^m is the solution to (4.7) then there is $C = C(d, p, K, \kappa, T)$ such that*

$$\mathbb{E}\sup_{0\le t\le T}|u^m_t|^p_{L^p} + \mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u^m_s|^2|u^m_s|^{p-2}\,dxds \le C\mathbb{E}\mathscr{K}_T. \tag{4.13}$$

Proof. From (4.10) and Remark 4.3, we get

$$\begin{aligned}\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_t)dx + \frac{\kappa}{2}\mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u^m_s|^2\phi''_n(u^m_s)dxds \le C\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_0)dx \\ + \mathbb{E}\int_0^t\int_{\mathscr{D}}\left[f^m_s(u^m_s,\nabla u^m_s)\phi'_n(u^m_s) + |f^0_s|^p\right]dxds + C\mathbb{E}\int_0^t\int_{\mathscr{D}}\left[|g_s|^p_{\ell^2} + \phi_n(u^m_s)\right]dxds,\end{aligned}$$

where $C = C(d, p, K, \kappa)$.

Taking limit $n\to\infty$ and using Lebesgue's dominated convergence theorem in view of (4.8), (4.9) and (4.12), we get

$$\begin{aligned}&\mathbb{E}\int_{\mathscr{D}}|u^m_t|^p dx + p(p-1)\frac{\kappa}{2}\mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u^m_s|^2|u^m_s|^{p-2}dxds \\ &\le C\mathbb{E}\mathscr{K}_t + p\mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^{p-2}f^m_s(u^m_s,\nabla u^m_s)u^m_s dxds + C\mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^p dxds.\end{aligned} \tag{4.14}$$

Using the fact $rf^m_t(r,0)\le 0$ for any $r\in\mathbb{R}, m\in\mathbb{N}, t\in[0,T]$, Young's inequality and Assumption A-4.3, we get

$$\begin{aligned}&p\mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^{p-2}f^m_s(u^m_s,\nabla u^m_s)u^m_s dxds \\ &\qquad = p\mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^{p-2}\big[f^m_s(u^m_s,\nabla u^m_s) - f^m_s(u^m_s,0) + f^m_s(u^m_s,0)\big]u^m_s dxds \\ &\qquad \le \mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^{p-2}\big[\frac{\kappa}{4}|f^m_s(u^m_s,\nabla u^m_s) - f^m_s(u^m_s,0)|^2 + C|u^m_s|^2\big]dxds \\ &\qquad \le \frac{\kappa}{4}\mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^{p-2}|\nabla u^m_s|^2 dxds + C\mathbb{E}\int_0^t\int_{\mathscr{D}}|u^m_s|^p dxds\end{aligned}$$

Substituting this in (4.14) and then applying Gronwall's lemma, we obtain for any $t\in[0,T]$

$$\mathbb{E}\int_{\mathscr{D}}|u^m_t|^p dx + \mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u^m_s|^2|u^m_s|^{p-2}dxds \le C\mathbb{E}\mathscr{K}_t \tag{4.15}$$

where $C = C(d, p, K, \kappa, T)$. Further, taking the supremum over $t\in[0,T]$ in (4.10), using the same estimates as given above and then taking expectation, we get using (4.11)

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx \le & C\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_0)dx + \mathbb{E}\sup_{0\le t\le T}\int_0^t\int_{\mathscr{D}}f^m_s(u^m_s,\nabla u^m_s)\phi'_n(u^m_s)dxds \\ &+ C\mathbb{E}\int_0^T\int_{\mathscr{D}}\left[|f^0_s|^p + |g_s|^p_{\ell^2} + \phi_n(u^m_s)\right]dxds \\ &+ \frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx + C\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u^m_s|^2\phi''_n(u^m_s)dxds,\end{aligned}$$

where C does not depend on n and m. Taking limit $n\to\infty$ using Lebesgue's dominated convergence theorem and using (4.15) along with the steps as above, we get

$$\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}|u^m_t|^p dx \le C\mathbb{E}\mathscr{K}_T + \frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}|u^m_t|^p dx$$

and hence the lemma. □

To complete the proof of Theorem 4.1 we need to take the limit, as $m \to \infty$ in (4.13) and to show that (4.1) has a solution. To that end we obtain the following result.

Lemma 4.4. *There is a subsequence of (m) denoted by (m') and an adapted process u such that $u \in L^\alpha(\Omega \times (0,T); L^\alpha(\mathscr{D})) \cap L^2(\Omega \times (0,T); W_0^{1,2}(\mathscr{D}))$ and almost surely $u \in C([0,T]; L^2(\mathscr{D}))$. Moreover, there exists $f' \in L^{\frac{\alpha}{\alpha-1}}\big(\Omega \times (0,T); L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big)$ such that*

$$
\begin{aligned}
u^m \rightharpoonup u \quad &in \quad L^\alpha(\Omega \times (0,T); L^\alpha(\mathscr{D})) \cap L^2(\Omega \times (0,T); W_0^{1,2}(\mathscr{D})),\\
f^m(u^m, \nabla u^m) \rightharpoonup f' \quad &in \quad L^{\frac{\alpha}{\alpha-1}}\big(\Omega \times (0,T); L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big),\\
L(u^m) \rightharpoonup L(u) \quad &in \quad L^2\big(\Omega \times (0,T); W^{-1,2}(\mathscr{D})\big)\\
M(u^m) \rightharpoonup M(u) \quad &in \quad L^2\big(\Omega \times (0,T); \ell^2(L^2(\mathscr{D}))\big).
\end{aligned}
$$

Finally for all $t \in [0,T]$,

$$
u_t = u_0 + \int_0^t (L_s u_s + f'_s + f^0_s) ds + \sum_{k \in \mathbb{N}} \int_0^t (M^k_s u_s + g^k_s) dW^k_s \ a.s.
$$

and

$$
\begin{aligned}
|u_t|^2_{L^2} =& |\phi|^2_{L^2} + 2\int_0^t \langle L_s u_s + f^0_s, u_s\rangle\, ds + 2\int_0^t \langle f'_s, u_s\rangle\, ds\\
&+ 2\sum_{k\in\mathbb{N}} \int_0^t (M^k_s u_s + g^k_s, u_s)\, dW^k_s + \sum_{k\in\mathbb{N}} \int_0^t |M^k_s u_s + g^k_s|^2_{L^2}\, ds\,.
\end{aligned}
$$

Proof. By Lemma 4.3, we have $u^m \in L^\alpha(\Omega\times(0,T); L^\alpha(\mathscr{D}))\cap L^2(\Omega\times(0,T); W_0^{1,2}(\mathscr{D}))$. Moreover, using Assumption A-4.3 and (4.13), we have

$$
\begin{aligned}
\mathbb{E}\int_0^T \int_{\mathscr{D}} |f^m_t(u^m_t(x), \nabla u^m_t(x))|^{\frac{\alpha}{\alpha-1}}\, dxdt &\le K\mathbb{E}\int_0^T \int_{\mathscr{D}} (1 + |u^m_t(x)|)^\alpha dxdt\\
&\le C + C\mathbb{E} \sup_{0\le t\le T} \int_{\mathscr{D}} |u^m_t(x)|^\alpha dx < \infty.
\end{aligned}
\tag{4.16}
$$

Thus, $f^m(u^m, \nabla u^m) \in L^{\frac{\alpha}{\alpha-1}}\big(\Omega \times (0,T); L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big)$ such that (4.13) and (4.16) holds for each $m \in \mathbb{N}$ with a constant independent of m. Since these Banach spaces are reflexive, there exists a subsequence (m') (see, e.g., Theorem 3.18 in [2]) such that

$$
\begin{aligned}
u^{m'} \rightharpoonup v \quad &\text{in} \quad L^\alpha(\Omega \times (0,T); L^\alpha(\mathscr{D}))\,,\\
u^{m'} \rightharpoonup \bar{v} \quad &\text{in} \quad L^2(\Omega \times (0,T); W_0^{1,2}(\mathscr{D})) \text{ and}\\
f^{m'}(u^{m'}, \nabla u^{m'}) \rightharpoonup f' \quad &\text{in} \quad L^{\frac{\alpha}{\alpha-1}}\big(\Omega \times (0,T); L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big)\,.
\end{aligned}
$$

Moreover, the operators L and M are bounded and linear and hence map a weakly convergent sequence to a weakly convergent sequence. Thus, we have

$$
\begin{aligned}
L(u^{m'}) \rightharpoonup L(\bar{v}) \quad &\text{in} \quad L^2\big(\Omega \times (0,T); W^{-1,2}(\mathscr{D})\big) \text{ and}\\
M(u^{m'}) \rightharpoonup M(\bar{v}) \quad &\text{in} \quad L^2\big(\Omega \times (0,T); \ell^2(L^2(\mathscr{D}))\big)\,.
\end{aligned}
$$

Note that for any adapted and bounded real valued process η_t and $\xi \in C_0^\infty(\mathscr{D})$, we have

$$
\mathbb{E}\int_0^T \eta_t \langle v_t - \bar{v}_t, \xi\rangle dt = \mathbb{E}\int_0^T \eta_t \langle v_t - u^{m'}_t, \xi\rangle dt + \mathbb{E}\int_0^T \eta_t \langle u^{m'}_t - \bar{v}_t, \xi\rangle dt \to 0
$$

as $m' \to \infty$. Since $C_0^\infty(\mathscr{D})$ is dense in $L^\alpha(\mathscr{D})$ and $W_0^{1,2}(\mathscr{D})$, we have the processes v and $\bar{v}$ are equal $dt \times \mathbb{P}$ almost everywhere. Further, the Bochner integral and the stochastic integral are bounded linear operators and hence are continuous with respect to weak topologies. Again, we have

$$\mathbb{E}\int_0^T \eta_t(u_t^{m'},\xi)dt$$
$$= \mathbb{E}\int_0^T \eta_t\Big((u_0^{m'},\xi) + \int_0^t \langle L_s u_s^{m'} + f_s^{m'} + f_s^0, \xi\rangle ds + \sum_{k\in\mathbb{N}}\int_0^t (\xi, M_s^k u_s^{m'} + g_s^k)dW_s^k\Big)dt.$$

On taking limit $m' \to \infty$, we get

$$\mathbb{E}\int_0^T \eta_t(v_t,\xi)dt = \mathbb{E}\int_0^T \eta_t\Big((u_0,\xi) + \int_0^t \langle L_s v_s + f_s' + f_s^0, \xi\rangle ds + \sum_{k\in\mathbb{N}}\int_0^t (\xi, M_s^k v_s + g_s^k)dW_s^k\Big)dt$$

for any adapted and bounded real valued process η_t and $\xi \in C_0^\infty(\mathscr{D})$. Since $C_0^\infty(\mathscr{D})$ is dense in $L^\alpha(\mathscr{D})$ and $W_0^{1,2}(\mathscr{D})$, we have

$$v_t = u_0 + \int_0^t (L_s v_s + f_s' + f_s^0)ds + \sum_{k\in\mathbb{N}}\int_0^t (M_s^k v_s + g_s^k)dW_s^k$$

$dt\times\mathbb{P}$ almost everywhere. Using Itô's formula for processes taking values in intersection of Banach spaces from Gyöngy and Šiška [16], there exists an $L^2(\mathscr{D})$-valued continuous modification u of v which satisfies above equality almost surely for all $t\in[0,T]$. ☐

Remark 4.4. For $\psi \in L^\alpha(\Omega\times(0,T);L^\alpha(\mathscr{D}))\cap L^2(\Omega\times(0,T);W_0^{1,2}(\mathscr{D}))$, we have

$$f^{m'}(\psi,\nabla\psi)\to f(\psi,\nabla\psi)$$

in $L^{\frac{\alpha}{\alpha-1}}(\Omega\times(0,T);L^{\frac{\alpha}{\alpha-1}}(\mathscr{D}))$. Indeed, by definition of $f^{m'}$, as $m'\to\infty$

$$f_s^{m'}(\psi_s(x),\nabla\psi_s(x)) \to f_s(\psi_s(x),\nabla\psi_s(x)) \;\; \forall \omega, s, x\,.$$

Moreover $|f_s^{m'}(r,z)| \le |f_s(r,z)|$ and due to Assumption A-4.3,

$$\mathbb{E}\int_0^T |f_s(\psi_s,\nabla\psi_s(x))|^{\frac{\alpha}{\alpha-1}}_{L^{\frac{\alpha}{\alpha-1}}}ds \le C\mathbb{E}\int_0^T\int_{\mathscr{D}}\Big(1+|\psi_s(x)|^\alpha\Big)dxds < \infty.$$

Therefore we may use Lebesgue Dominated Convergence Theorem to obtain

$$\lim_{m'\to\infty}\mathbb{E}\int_0^T\int_{\mathscr{D}} |f_s^{m'}(\psi_s(x),\nabla\psi_s(x)) - f_s(\psi_s(x),\nabla\psi_s(x))|^{\frac{\alpha}{\alpha-1}}dxds$$
$$= \mathbb{E}\int_0^T\int_{\mathscr{D}}\lim_{m'\to\infty}|f_s^{m'}(\psi_s(x),\nabla\psi_s(x)) - f_s(\psi_s(x),\nabla\psi_s(x))|^{\frac{\alpha}{\alpha-1}}dxds = 0.$$

Proof of Theorem 4.1. In order to show the weak limit u obtained in Lemma 4.4 is indeed the unique solution of SPDE (4.1), it remains to show that $f' = f(u,\nabla u)$ which can be shown using the monotonicity argument as below.

Define for each $w\in L^\alpha(\mathscr{D})\cap W_0^{1,2}(\mathscr{D})$, $s\in(0,T)$ and $k\in\mathbb{N}$, the operators

$$A_s w := L_s w + f_s^0 \quad\text{and}\quad B_s^k w := M_s^k w + g_s^k.$$

Then for any $w,w'\in L^\alpha(\mathscr{D})\cap W_0^{1,2}(\mathscr{D})$, we have using Remark 4.2

$$2\langle A_s w - A_s w', w-w'\rangle + \sum_{k\in\mathbb{N}}|B_s^k w - B_s^k w'|^2_{L^2} \le -\kappa|w-w'|^2_{W_0^{1,2}} + K'|w-w'|^2_{L^2}. \tag{4.17}$$

Consider $\psi\in L^\alpha(\Omega\times(0,T);L^\alpha(\mathscr{D}))\cap L^2(\Omega\times(0,T);W_0^{1,2}(\mathscr{D}))$. Then using Assumption A-4.3, Remark 4.1 and definition of f^m, we have

$$\langle f_s^{m'}(u_s^{m'},\nabla u_s^{m'}) - f_s^{m'}(\psi_s,\nabla u_s^{m'}), u_s^{m'}-\psi_s\rangle \le 0 \tag{4.18}$$

almost surely for all $s \in [0,T]$. Moreover using Young's inequality and Assumption A-4.3, we have almost surely for all $s \in [0,T]$,

$$2\langle f_s^{m'}(\psi_s, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla \psi_s), u_s^{m'} - \psi_s\rangle \le \kappa |\nabla(u_s^{m'} - \psi_s)|_{L^2}^2 + C|u_s^{m'} - \psi_s|_{L^2}^2. \tag{4.19}$$

Define $K'' := K' + C$, where K' and C are as in (4.17) and (4.19) above. Then using the product rule and Itô's formula, we obtain

$$\begin{aligned}&\mathbb{E}\big(e^{-K''t}|u_t|_{L^2}^2\big) - \mathbb{E}(|u_0|_{L^2}^2)\\ &\qquad = \mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s u_s + f_s', u_s\rangle + \sum_{k\in\mathbb{N}} |B_s^k u_s|_{L^2}^2 - K''|u_s|_{L^2}^2\Big) ds\Big]\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\mathbb{E}\big(e^{-K''t}|u_t^{m'}|_{L^2}^2\big) - \mathbb{E}(|u_0^{m'}|_{L^2}^2) = \mathbb{E}\Big[\int_0^t e^{-K''s}\Big(&2\langle A_s u_s^{m'} + f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}), u_s^{m'}\rangle\\ &+ \sum_{k\in\mathbb{N}} |B_s^k u_s^{m'}|_{L^2}^2 - K''|u_s^{m'}|_{L^2}^2\Big) ds\Big]\end{aligned} \tag{4.21}$$

for all $t \in [0,T]$. We now need to re-arrange the right-hand side of (4.21) so that we can use the monotonicity assumptions. We have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s u_s^{m'} + f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}), u_s^{m'}\rangle + \sum_{k\in\mathbb{N}} |B_s^k u_s^{m'}|_{L^2}^2 - K''|u_s^{m'}|_{L^2}^2\Big) ds\Big]\\ =&\mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s u_s^{m'} - A_s\psi_s, u_s^{m'} - \psi_s\rangle + 2\langle A_s\psi_s, u_s^{m'}\rangle + 2\langle A_s u_s^{m'} - A_s\psi_s, \psi_s\rangle\\ &+ 2\langle f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla\psi_s), u_s^{m'} - \psi_s\rangle + 2\langle f_s^{m'}(\psi_s, \nabla\psi_s), u_s^{m'}\rangle\\ &+ 2\langle f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla\psi_s), \psi_s\rangle + \sum_{k\in\mathbb{N}} \big|B_s^k u_s^{m'} - B_s^k\psi_s\big|_{L^2}^2 - \sum_{k\in\mathbb{N}} |B_s^k\psi_s|_{L^2}^2\\ &+ 2\sum_{k\in\mathbb{N}} \big(B_s^k u_s^{m'}, B_s^k\psi_s\big) - K''\Big[|u_s^{m'} - \psi_s|_{L^2}^2 - |\psi_s|_{L^2}^2 + 2(u_s^{m'}, \psi_s)\Big]\Big) ds\Big].\end{aligned} \tag{4.22}$$

Using (4.18) and (4.19), we have

$$\begin{aligned}&2\langle f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla\psi_s), u_s^{m'} - \psi_s\rangle\\ &= 2\langle f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla u_s^{m'}) + f_s^{m'}(\psi_s, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla\psi_s), u_s^{m'} - \psi_s\rangle\\ &\le \kappa|\nabla(u_s^{m'} - \psi_s)|_{L^2}^2 + C|u_s^{m'} - \psi_s|_{L^2}^2\end{aligned}$$

and hence using (4.17) in (4.22) together with (4.21), we obtain for all $t \in [0,T]$

$$\begin{aligned}\mathbb{E}\big(e^{-K''t}|u_t^{m'}|_{L^2}^2\big) - \mathbb{E}(|u_0^{m'}|_{L^2}^2) \le \mathbb{E}\Big[\int_0^t e^{-K''s}\Big(&2\langle A_s\psi_s, u_s^{m'}\rangle + 2\langle A_s u_s^{m'} - A_s\psi_s, \psi_s\rangle\\ &+ 2\langle f_s^{m'}(\psi_s, \nabla\psi_s), u_s^{m'}\rangle + 2\langle f_s^{m'}(u_s^{m'}, \nabla u_s^{m'}) - f_s^{m'}(\psi_s, \nabla\psi_s), \psi_s\rangle\\ &- \sum_{k\in\mathbb{N}} |B_s^k\psi_s|_{L^2}^2 + 2\sum_{k\in\mathbb{N}} \big(B_s^k u_s^{m'}, B_s^k\psi_s\big) + K''\big[|\psi_s|_{L^2}^2 - 2(u_s^{m'}, \psi_s)\big]\Big) ds\Big].\end{aligned}$$

Now, integrating over t from 0 to T, letting $m \to \infty$ and using the weak lower semicontinuity of the norm, we obtain

$$\mathbb{E}\Big[\int_0^T \big(e^{-K''t}|u_t|_{L^2}^2 - |u_0|_{L^2}^2\big) dt\Big] \le \liminf_{m'\to\infty} \mathbb{E}\Big[\int_0^T \big(e^{-K''t}|u_t^{m'}|_{L^2}^2 - |u_0^{m'}|_{L^2}^2\big) dt\Big]$$

$$
\begin{aligned}
\leq & \mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(2\langle A_s\psi_s, u_s\rangle + 2\langle A_s u_s - A_s\psi_s, \psi_s\rangle \\
& + 2\langle f_s(\psi_s, \nabla\psi_s), u_s\rangle + 2\langle f'_s - f_s(\psi_s, \nabla\psi_s), \psi_s\rangle - \sum_{k\in\mathbb{N}} |B^k_s\psi_s|^2_{L^2} \\
& + 2\sum_{k\in\mathbb{N}}(B^k_s u_s, B^k_s(\psi_s)) + K''\left[|\psi_s|^2_{L^2} - 2(u_s,\psi_s)\right]\Big)dsdt\Big]
\end{aligned} \tag{4.23}
$$

where we have used Remark 4.4 in last inequality. Again, integrating from 0 to T in (4.20) and combining this with (4.23), we get

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(& 2\langle A_s u_s - A_s\psi_s, u_s - \psi_s\rangle + 2\langle f'_s - f_s(\psi_s, \nabla\psi_s), u_s - \psi_s\rangle \\
& + \sum_{k\in\mathbb{N}} |B^k_s\psi_s - B^k_s u_s|^2_{L^2} - K''|u_s - \psi_s|^2_{L^2}\Big)dsdt\Big] \leq 0
\end{aligned}
$$

which on using (4.17) gives

$$
\mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(2\langle f'_s - f_s(\psi_s, \nabla\psi_s), u_s - \psi_s\rangle - C|u_s - \psi_s|^2_{L^2}\Big)dsdt\Big] \leq 0. \tag{4.24}
$$

Let $\eta \in L^\infty((0,T)\times\Omega;\mathbb{R})$, $\phi \in C_0^\infty(\mathscr{D})$, $\epsilon \in (0,1)$ and let $\psi = u - \epsilon\eta\phi$. Then from (4.24) we obtain that,

$$
\mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(2\epsilon\langle f'_s - f_s(u_s - \epsilon\eta_s\phi, \nabla u_s - \epsilon\eta_s\nabla\phi), \eta_s\phi\rangle - C\epsilon^2|\eta_s\phi|^2_{L^2}\Big)dsdt\Big] \leq 0.
$$

Dividing by ϵ, letting $\epsilon \to 0$, using Lebesgue dominated convergence theorem and Assumption A-4.3 leads to

$$
\mathbb{E}\Big[\int_0^T\int_0^t 2e^{-K''s}\eta_s\langle f'_s - f_s(u_s, \nabla u_s), \phi\rangle dsdt\Big] \leq 0.
$$

Since this holds for any $\eta \in L^\infty(\Omega\times(0,T);\mathbb{R})$ and $\phi \in C_0^\infty(\mathscr{D})$, we get that $f(u,\nabla u) = f'$ which concludes the proof.

Further, taking $m \to \infty$ in (4.13) and using the weak lower semicontinuity of the norm, we obtain the following estimates for the solution of (4.1)

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq t\leq T}|u_t|^p_{L^p} + \mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u_s|^2|u_s|^{p-2}dxds \\
& \leq \liminf_{m\to\infty}\Big[\mathbb{E}\sup_{0\leq t\leq T}|u^m_t|^p_{L^p} + \mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u^m_s|^2|u^m_s|^{p-2}dxds\Big] \leq C\mathbb{E}\Big(|\phi|^p_{L^p} + \|f^0\|^p_{L^p} + \||g|_{\ell^2}\|^p_{L^p}\Big)
\end{aligned}
$$

as desired. □

4.2 Interior regularity

In this section, we present the results on interior regularity of the solution to SPDE (4.1). The main result is stated in Theorem 4.2. The idea is to prove the result for the linear SPDE first and then use it along with the L^p-estimates obtained in Section 4.1 to prove Theorem 4.2. We do not claim the result for the linear case to be new, however we could not find such result in literature in sufficient generality.

To raise the regularity of the solution one needs the given data to be sufficiently smooth. Thus, we assume the following condition on the coefficients before stating the main result of this section.

Let $n \geq 0$ be an integer.

A - 4.5. For any $i, j = 1, \ldots, d$, the coefficients a^{ij}, b^i and c and their spatial derivatives up to order n are real-valued, $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and are bounded by K. The coefficients $\sigma^i = (\sigma^{ik})_{k=1}^{\infty}$, $\mu = (\mu^k)_{k=1}^{\infty}$ and their spatial derivatives up to order n are ℓ^2-valued, $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and almost surely,

$$\sum_{i=1}^{d} \sum_{k \in \mathbb{N}} \sum_{|\gamma| \leq n} |D^\gamma \sigma_t^{ik}(x)|^2 + \sum_{k \in \mathbb{N}} \sum_{|\gamma| \leq n} |D^\gamma \mu_t^k(x)|^2 \leq K$$

for all t and x.

Theorem 4.2. *Let Assumptions A-4.2 to A-4.4 hold and u be the solution to* (4.1)*. Fix some open $\mathscr{D}' \Subset \mathscr{D}$.*

(i) If Assumption A-4.5 holds with $n = 1$, and if $\phi \in L^2(\Omega; W^{1,2}(\mathscr{D}))$ and $g \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}; \ell^2))$, then

$$u \in C([0,T]; W^{1,2}(\mathscr{D}')) \text{ a.s. and } u \in L^2(\Omega \times (0,T); W^{2,2}(\mathscr{D}')).$$

(ii) Moreover, in case the semilinear term f does not depend on z, if Assumption A-4.5 holds with $n = 2$, if $\phi \in L^2(\Omega; W^{2,2}(\mathscr{D}))$, $f^0 \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}))$ and $g \in L^2(\Omega \times (0,T); W^{2,2}(\mathscr{D}; \ell^2))$ and if almost surely

$$|\partial_r f_t(x,r)| \leq K(1 + |r|)^{\alpha-2} \text{ and } |\partial_i f_t(x,r)| \leq K(1+|r|)^{\alpha-1} \tag{4.25}$$

for all $i = 1, \ldots, d$, $t \in [0,T]$, $x \in \mathscr{D}$ and all $r \in \mathbb{R}$, then we have

$$u \in C([0,T]; W^{2,2}(\mathscr{D}')) \text{ a.s. and } u \in L^2(\Omega \times (0,T); W^{3,2}(\mathscr{D}')).$$

One can obtain regularity results up to the boundary in appropriate weighted Sobolev space using results from Krylov [24] along with the L^p-estimates obtained in Theorem 4.1. However, obtaining the similar results for the linear equations using L^p-theory is more useful . We will discuss this in Section 4.3.

As mentioned before, we will first get the results for linear equations. So, we consider the following linear stochastic evolution equation:

$$dv_t = (L_t v_t + f_t)dt + \sum_{k \in \mathbb{N}} (M_t^k v_t + g_t^k) dW_t^k \quad \text{on } [0,T] \times \mathscr{D}, \tag{4.26}$$

where the operators L and M^k are defined in (4.2). As can be seen in what follows, one can raise the regularity to any order for the linear equation by assuming the given data to be sufficiently smooth. Thus we make the following assumption on initial data and the free terms and then state the result in Theorem 4.3.

A - 4.6. Assume that $v_0 \in L^2(\Omega; W^{n,2}(\mathscr{D}))$, $g \in L^2(\Omega \times (0,T); W^{n,2}(\mathscr{D}; \ell^2))$ and $f \in L^2(\Omega \times (0,T); W^{n-1,2}(\mathscr{D}))$.

Theorem 4.3. *Assume that v is a continuous $L^2(\mathscr{D})$ - valued adapted process such that $v \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}))$, and it satisfies* (4.26)*. If Assumptions A- 4.2, A- 4.5 and A- 4.6 hold, then for all open $\mathscr{D}' \Subset \mathscr{D}$,*

$$v \in C([0,T]; W^{n,2}(\mathscr{D}')) \text{ a.s. and } v \in L^2(\Omega \times (0,T); W^{n+1,2}(\mathscr{D}'))$$

We will prove Theorem 4.3 via Lemmas 4.5 and 4.6. In Lemma 4.5, we first prove the special case $n = 1$.

Lemma 4.5. *Assume that $v \in C([0,T]; L^2(\mathscr{D}))$ a.s., v is adapted and satisfies* (4.26) *and moreover $v \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}))$. If Assumptions A-4.2, A-4.5 and A-4.6 hold with $n = 1$, then*

$$\mathbb{E} \sup_{0 \leq t \leq T} |\partial_i v_t|^2_{L^2(\mathscr{D}')} + \mathbb{E} \int_0^T |\partial_i v_t|^2_{W^{1,2}(\mathscr{D}')} dt$$

$$\leq C\left[\mathbb{E}\int_{\mathscr{D}}|\nabla v_0|^2dx+\mathbb{E}\int_0^T\int_{\mathscr{D}}\Big[|\nabla v_t|^2+|f_t|^2+|v_t|^2+\sum_{k\in\mathbb{N}}|\nabla g_t^k|^2\Big]dxdt\right] \tag{4.27}$$

for all $i=1,\dots,d$ and open $\mathscr{D}'\Subset\mathscr{D}$ where $C=C(\mathscr{D}',d,T,K,\kappa)$.

Proof. We consider a cut-off function $\eta\in C_0^\infty(\mathscr{D})$ which is 1 on $\mathscr{D}'$. Define the l^{th}-difference quotient, $l\in\{1,2,\dots,d\}$, by

$$\delta_l^h u(x):=\frac{1}{h}\big(T_l^h u-u\big)(x),\qquad x\in\mathbb{R}^d$$

where $T_l^h u(x)=u(x+he_l)$ is the shift operator and the step-size h satisfies $2|h|<\operatorname{dist}(\operatorname{supp}\eta,\partial\mathscr{D})$. From (4.26), we get [1]

$$d(\eta\delta_l^h v_t)=\eta\delta_l^h(L_tv_t+f_t)dt+\eta\sum_{k\in\mathbb{N}}\delta_l^h(M_t^kv_t+g_t^k)dW_t^k. \tag{4.30}$$

Applying Itô's formula for the square of L^2-norm, we get

$$\begin{aligned}d|\eta\delta_l^h v_t|^2_{L^2(\mathscr{D})}=2\langle\eta\delta_l^h(L_tv_t+f_t),\eta\delta_l^hv_t\rangle dt+2\sum_{k\in\mathbb{N}}(\eta\delta_l^h(M_t^kv_t+g_t^k),\eta\delta_l^hv_t)dW_t^k\\+\sum_{k\in\mathbb{N}}|\eta\delta_l^h(M_t^kv_t+g_t^k)|^2_{L^2}dt,\end{aligned}$$

that is,

$$\begin{aligned}\int_{\mathscr{D}}\eta^2|\delta_l^hv_t|^2dx&=\int_{\mathscr{D}}\eta^2|\delta_l^hv_0|^2dx+2\int_0^t\int_{\mathscr{D}}\eta^2\delta_l^h(L_sv_s+f_s)\delta_l^hv_sdxds\\&+2\sum_{k\in\mathbb{N}}\int_0^t\int_{\mathscr{D}}\eta^2\delta_l^h(M_s^kv_s+g_s^k)\delta_l^hv_sdxdW_s^k+\sum_{k\in\mathbb{N}}\int_0^t\int_{\mathscr{D}}\eta^2|\delta_l^h(M_s^kv_s+g_s^k)|^2dxds.\end{aligned}$$

It follows from the definition of δ_l^h and linearity of ∂_j, that the two operators commute. Thus, using integration by parts and the formula,

$$\delta_l^h(vw)(x)=\delta_l^hv(x)T_l^hw(x)+v(x)\delta_l^hw(x)$$

we get,

$$\int_{\mathscr{D}}\eta^2|\delta_l^hv_t|^2dx=I_0-2\int_0^t\int_{\mathscr{D}}\eta^2\sum_{i,j=1}^d a_s^{ij}\,\partial_i(\delta_l^hv_s)\,\partial_j(\delta_l^hv_s)+I_1+I_2+I_3+\mathscr{M}_t^h+I_4 \tag{4.31}$$

[1] Note that (4.26) does not hold pointwise. Thus (4.30) is obtained from (4.26) with the help of test functions $\frac{1}{h}\eta\phi$ and $\frac{1}{h}T_l^{-1}(\eta\phi)$, $\phi\in C_0^\infty(\mathscr{D})$. Indeed we have from (4.26),

$$d\big(v_t,\frac{1}{h}\eta\phi\big)=\big\langle L_tv_t+f_t,\frac{1}{h}\eta\phi\big\rangle dt+\sum_{k\in\mathbb{N}}\big(M_t^kv_t+g_t^k,\frac{1}{h}\eta\phi\big)dW_t^k$$

which can be rewritten as

$$d\big(\frac{1}{h}\eta v_t,\phi\big)=\big\langle\frac{1}{h}\eta(L_tv_t+f_t),\phi\big\rangle dt+\sum_{k\in\mathbb{N}}\big(\frac{1}{h}\eta(M_t^kv_t+g_t^k),\phi\big)dW_t^k\,. \tag{4.28}$$

Similarly, taking the test function $\frac{1}{h}T_l^{-h}(\eta\phi)$ and observing that integrals on $\mathscr{D}$ in this case are same as integrals on $\mathbb{R}^d$ (since for our choice of h, $\operatorname{supp}\eta\cup\operatorname{supp}(T_l^h\eta)\cup\operatorname{supp}(T_l^{-h}\eta)\Subset\mathscr{D}$), we obtain with the help of change of variable that

$$d\big(\frac{1}{h}\eta T_l^hv_t,\phi\big)=\big\langle\frac{1}{h}\eta T_l^h(L_tv_t+f_t),\phi\big\rangle dt+\sum_{k\in\mathbb{N}}\big(\frac{1}{h}\eta T_l^h(M_t^kv_t+g_t^k),\phi\big)dW_t^k\,. \tag{4.29}$$

Subtracting (4.28) from (4.29), we obtain (4.30) as desired.

where,

$$I_0 := \int_{\mathscr{D}} \eta^2 |\delta_l^h v_0|^2 dx\,, \quad I_1 := -2\int_0^t \int_{\mathscr{D}} \eta^2 \sum_{i,j=1}^d \delta_l^h a_s^{ij}\, \partial_i(T_l^h v_s)\partial_j(\delta_l^h v_s) dxds\,,$$

$$I_2 := -4\int_0^t \int_{\mathscr{D}} \eta \sum_{i,j=1}^d \left[\delta_l^h a_s^{ij}\, \partial_i(T_l^h v_s) + a_s^{ij}\, \partial_i(\delta_l^h v_s)\right]\partial_j \eta \delta_l^h v_s dxds\,,$$

$$I_3 := 2\int_0^t \int_{\mathscr{D}} \eta^2 \Big[\sum_{i=1}^d \{\delta_l^h b_s^i\, \partial_i(T_l^h v_s) + b_s^i\, \delta_l^h(\partial_i v_s)\} + \delta_l^h c_s\, T_l^h v_s + c_s\, \delta_l^h v_s + \delta_l^h f_s\Big]\delta_l^h v_s dxds\,,$$

$$I_4 := \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \Big|\sum_{i=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s) + \delta_l^h \mu_s^k\, T_l^h v_s + \sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s) + \mu_s^k\, \delta_l^h v_s + \delta_l^h g_s^k\Big|^2 dxds$$

$$\text{and} \quad \mathscr{M}_t^h := 2\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \delta_l^h(M_s^k v_s + g_s^k)\delta_l^h v_s dxdW_s^k.$$

Now, we see that

$$\begin{aligned}
I_4 =& \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \Bigg[\Big|\sum_{i=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s) + \delta_l^h \mu_s^k\, T_l^h v_s\Big|^2 + \Big|\sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s) + \mu_s^k\, \delta_l^h v_s + \delta_l^h g_s^k\Big|^2 \\
&+ 2\Big[\sum_{i=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s) + \delta_l^h \mu_s^k\, T_l^h v_s\Big]\Big[\sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s) + \mu_s^k \delta_l^h v_s + \delta_l^h g_s^k\Big]\Bigg] dxds \\
\leq& \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \Big[\sum_{i,j=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s)\, \sigma_s^{jk}\, \partial_j(\delta_l^h v_s)\Big] dxds + \bar{I}_4
\end{aligned}$$

where,

$$\begin{aligned}
\bar{I}_4 :=& \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \Bigg[(d+1)\sum_{i=1}^d |\delta_l^h \sigma_s^{ik}|^2 |\partial_i(T_l^h v_s)|^2 + (d+1)|\delta_l^h \mu_s^k\, T_l^h v_s|^2 \\
&+ 2\sum_{i,j=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s)\, \sigma_s^{jk}\, \partial_j(\delta_l^h v_s) + 2\sum_{i,j=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s)\, \mu_s^k\, \delta_l^h v_s \\
&+ 2\sum_{i,j=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s)\, \delta_l^h g_s^k + 2\sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s)\, \delta_l^h \mu_s^k\, T_l^h v_s + 2\delta_l^h \mu_s^k\, T_l^h v_s\, \mu_s^k\, \delta_l^h v_s \\
&+ 2\delta_l^h \mu_s^k\, T_l^h v_s\, \delta_l^h g_s^k + |\mu_s^k\, \delta_l^h v_s|^2 + |\delta_l^h g_s^k|^2 + 2\sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s)\, \mu_s^k\, \delta_l^h v_s \\
&+ 2\sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s)\, \delta_l^h g_s^k + 2\mu_s^k\, \delta_l^h v_s\, \delta_l^h g_s^k\Bigg] dxds
\end{aligned}$$

Substituting this in (4.31), we get

$$\begin{aligned}
\int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx \leq& I_0 + I_1 - 2\int_0^t \int_{\mathscr{D}} \eta^2 \sum_{i,j=1}^d \Big[a_s^{ij} - \frac{1}{2}\sum_{k\in\mathbb{N}} \sigma_s^{ik}\sigma_s^{jk}\Big]\partial_i(\delta_l^h v_s)\, \partial_j(\delta_l^h v_s) dxds \\
&+ I_2 + I_3 + \mathscr{M}_t^h + \bar{I}_4.
\end{aligned}$$

which on using Assumptions A-4.2, A-4.5 (with $n = 1$) and Young's inequality for an $\epsilon > 0$ gives,

$$\int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx \leq \int_{\mathscr{D}} \eta^2 |\delta_l^h v_0|^2 dx - 2\kappa \int_0^t \int_{\mathscr{D}} \eta^2 |\nabla(\delta_l^h v_s)|^2 dxds + \mathscr{M}_t^h$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathscr{D}} \sum_{i,j=1}^{d} \big[\epsilon K|\partial_i(T_l^h v_s)|^2 + \epsilon K|\partial_i(\delta_l^h v_s)|^2 + C_\epsilon|\delta_l^h v_s|^2\big]\eta\,\partial_j\eta\,dxds \\
&+\int_0^t \int_{\mathscr{D}} \eta^2 \Bigg[2\delta_l^h f_s\,\delta_l^h v_s + C_{K,d,\epsilon}\sum_{i=1}^{d}|\partial_i(T_l^h v_s)|^2 + C_{K,d,\epsilon}|T_l^h v_s|^2 \\
&+C\sum_{k\in\mathbb{N}}|\delta_l^h g_s^k|^2 + \epsilon C_K\sum_{i=1}^{d}|\partial_i(\delta_l^h v_s)|^2 + C_{K,\epsilon}|\delta_l^h v_s|^2\Bigg]\,dxds.
\end{aligned}
\tag{4.32}
$$

Now extending η, f, g and v to $\mathbb{R}^d$ by setting them to 0 on $\mathbb{R}^d \setminus \mathscr{D}$ and using the fact that $\operatorname{supp}\eta \subset \mathscr{D}$ and $\operatorname{supp}(T_l^{-h}\eta) \subset \mathscr{D}$ for our choice of h, we get

$$
\begin{aligned}
\int_{\mathscr{D}} \eta^2\,\delta_l^h f_s\,\delta_l^h v_s dx &= \int_{\mathbb{R}^d} \eta^2\,\delta_l^h f_s\,\delta_l^h v_s dx \\
&= \int_{\mathbb{R}^d} \eta^2\,\frac{1}{h}T_l^h f_s\,\delta_l^h v_s dx - \int_{\mathbb{R}^d} \eta^2\,\frac{1}{h}f_s\,\delta_l^h v_s dx \\
&= \int_{\mathbb{R}^d} T_l^{-h}(\eta^2)\frac{1}{h}f_s\,T_l^{-h}(\delta_l^h v_s)dx - \int_{\mathbb{R}^d} \eta^2\,\frac{1}{h}f_s\,\delta_l^h v_s dx \\
&= \int_{\mathbb{R}^d} f_s\frac{1}{h}\big[T_l^{-h}(\eta^2\delta_l^h v_s) - (\eta^2\delta_l^h v_s)\big]dx \\
&= -\int_{\mathbb{R}^d} f_s\,\delta_l^{-h}(\eta^2\,\delta_l^h v_s)dx \\
&= -\int_{\mathscr{D}} f_s\,\delta_l^{-h}(\eta^2\,\delta_l^h v_s)dx \\
&\le \epsilon\int_{\mathscr{D}} |\delta_l^{-h}(\eta^2\,\delta_l^h v_s)|^2dx + C_\epsilon\int_{\mathscr{D}}|f_s|^2dx
\end{aligned}
\tag{4.33}
$$

where last inequality has been obtained using Young's inequality. Since $\eta^2\,\delta_l^h v_s \in W^{1,2}(\mathscr{D})$, using the relation between difference quotients and weak derivatives (see e.g. [7, Ch. 5, Sec. 8, Theorem 3]), we have

$$
\int_{\mathscr{D}} |\delta_l^{-h}(\eta^2\,\delta_l^h v_s)|^2dx = \int_{\mathscr{D}_l^h(\eta)} |\delta_l^{-h}(\eta^2\,\delta_l^h v_s)|^2dx \le C\int_{\mathscr{D}}|\nabla(\eta^2\,\delta_l^h v_s)|^2dx
$$

for some constant C and $\mathscr{D}_l^h(\eta) := \operatorname{supp}\eta \cup \operatorname{supp}(T_l^h\eta) \cup \operatorname{supp}(T_l^{-h}\eta) \Subset \mathscr{D}$. Substituting this in (4.33), we get

$$
\begin{aligned}
&\int_{\mathscr{D}} \eta^2\,\delta_l^h f_s\,\delta_l^h v_s dx \\
&\le \epsilon C\int_{\mathscr{D}}|\nabla(\eta^2\,\delta_l^h v_s)|^2dx + C_\epsilon\int_{\mathscr{D}}|f_s|^2dx \\
&= \epsilon C\int_{\mathscr{D}}|\eta^2\,\nabla(\delta_l^h v_s) + 2\eta\,\nabla\eta\,\delta_l^h v_s|^2dx + C_\epsilon\int_{\mathscr{D}}|f_s|^2dx \\
&\le \epsilon C_\eta\int_{\mathscr{D}}|\eta\,\nabla(\delta_l^h v_s)|^2dx + \epsilon C_\eta\int_{\mathscr{D}}|(\eta\,\delta_l^h v_s)|^2dx + C_\epsilon\int_{\mathscr{D}}|f_s|^2dx.
\end{aligned}
\tag{4.34}
$$

Similarly,

$$
\int_{\mathscr{D}}\eta^2|T_l^h v_s|^2dx = \int_{\mathscr{D}_l^h(\eta)}\eta^2|T_l^h v_s|^2dx = \int_{\mathscr{D}_l^h(\eta)}|T_l^{-h}\eta|^2|v_s|^2dx \le C_\eta\int_{\mathscr{D}}|v_s|^2dx
$$

and

$$
\sum_{i=1}^{d}\int_{\mathscr{D}}\eta^2|\partial_i(T_l^h v_s)|^2dx = \sum_{i=1}^{d}\int_{\mathscr{D}_l^h(\eta)}\eta^2|T_l^h(\partial_i v_s)|^2dx \le C_\eta\sum_{i=1}^{d}\int_{\mathscr{D}}|\partial_i v_s|^2dx = C_\eta\int_{\mathscr{D}}|\nabla v_s|^2dx.
$$

Using the assumption $g \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D};\ell^2))$ and the property of difference quotients mentioned above,

$$\sum_{k\in\mathbb{N}} \int_{\mathscr{D}} \eta^2 |\delta_l^h\, g_s^k|^2 dx = \sum_{k\in\mathbb{N}} \int_{\mathscr{D}_l^h(\eta)} \eta^2 |\delta_l^h g_s^k|^2 dx \le C_\eta \sum_{k\in\mathbb{N}} \int_{\mathscr{D}} |\nabla g_s^k|^2 dx.$$

Similarly, $v \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}))$ and the property of difference quotients imply

$$\int_{\mathscr{D}} \eta^2 |\delta_l^h v_s|^2 dx \le C_\eta \int_{\mathscr{D}} |\nabla v_s|^2 dx. \tag{4.35}$$

Substituting (4.34)-(4.35) in (4.32), we get

$$\begin{aligned}\int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx \le C_\eta \int_{\mathscr{D}} |\nabla v_0|^2 dx - 2\kappa \int_0^t \int_{\mathscr{D}} \eta^2 |\nabla(\delta_l^h v_s)|^2 dx ds + \mathscr{M}_t^h \\ + \int_0^t \int_{\mathscr{D}} \Big[C_{K,\epsilon,\eta,d} |\nabla v_s|^2 + \epsilon C_{K,\eta} |\eta \nabla(\delta_l^h v_s)|^2 + C_\epsilon |f_s|^2 \\ + C_{K,\epsilon,\eta,d} |v_s|^2 + C_\eta \sum_{k\in\mathbb{N}} |\nabla g_s^k|^2 \Big] dx ds.\end{aligned} \tag{4.36}$$

Further, it can be seen that the process $\mathscr{M}_t^h$ defined in (4.31) is a local martingale where a localizing sequence of stopping times converging to T as $n \to \infty$ is given by,

$$\tau_n := \inf\{t \in [0,T] : |\eta \delta_l^h v_s|_{L^2(\mathscr{D})}| > n\} \wedge T.$$

Thus, replacing t by $t \wedge \tau_n$ in (4.36), then taking expectation and choosing $\epsilon > 0$ small enough such that $2\kappa - \epsilon C_{K,\eta} = C_\kappa > 0$ and finally using Fatou's lemma, we get

$$\begin{aligned}&\mathbb{E} \int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx + C_\kappa \mathbb{E} \int_0^t \int_{\mathscr{D}} \eta^2 |\nabla(\delta_l^h v_s)|^2 dx ds \\ &\quad \le C_\eta \mathbb{E} \int_{\mathscr{D}} |\nabla v_0|^2 dx + \mathbb{E} \int_0^t \int_{\mathscr{D}} \Big[C_{K,\epsilon,\eta,d} |\nabla v_s|^2 + C_\epsilon |f_s|^2 + C_{K,\epsilon,\eta,d} |v_s|^2 \\ &\quad\quad + C_\eta \sum_{k\in\mathbb{N}} |\nabla g_s^k|^2 \Big] dx ds.\end{aligned} \tag{4.37}$$

Using the inequalities of Burkholder–Davis–Gundy, Hölder and Young together with the estimates obtained above, we get that

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le T} |\mathscr{M}_{t\wedge\tau_n}^h| &= \mathbb{E} \sup_{0\le t\le T} \Big| 2 \sum_{k\in\mathbb{N}} \int_0^{t\wedge\tau_n} \int_{\mathscr{D}} \eta^2 \delta_l^h (M_s^k v_s + g_s^k) \delta_l^h v_s dx dW_s^k \Big| \\ &\le 4\mathbb{E} \Big(\sum_{k\in\mathbb{N}} \int_0^{\tau_n} \Big| 2 \int_{\mathscr{D}} \eta^2 \delta_l^h (M_s^k v_s + g_s^k) \delta_l^h v_s dx \Big|^2 ds \Big)^{\frac{1}{2}} \\ &\le 8\mathbb{E} \Big(\sum_{k\in\mathbb{N}} \int_0^{\tau_n} |\eta\, \delta_l^h (M_s^k v_s + g_s^k)|_{L^2(\mathscr{D})}^2 |\eta\, \delta_l^h v_s|_{L^2(\mathscr{D})}^2 ds \Big)^{\frac{1}{2}} \\ &\le \frac{1}{2} \mathbb{E} \sup_{0\le t\le T} |\eta\, \delta_l^h v_t|_{L^2(\mathscr{D})}^2 + C \sum_{k\in\mathbb{N}} \mathbb{E} \int_0^{\tau_n} |\eta\, \delta_l^h (M_s^k v_s + g_s^k)|_{L^2(\mathscr{D})}^2 ds \\ &\le \frac{1}{2} \mathbb{E} \sup_{0\le t\le T} |\eta\, \delta_l^h v_t|_{L^2(\mathscr{D})}^2 + C\mathbb{E} \int_0^{\tau_n} \int_{\mathscr{D}} \big[|\nabla v_s|^2 + |f_s|^2 + |v_s|^2 + |\nabla g_s|_{\ell^2}^2 \big] dx ds.\end{aligned} \tag{4.38}$$

Replacing t by $t \wedge \tau_n$ in (4.36), taking the supremum over $t \in [0,T]$ and using (4.38) we obtain,

$$\mathbb{E} \sup_{0\le t\le T} \int_{\mathscr{D}} \eta^2 |\delta_l^h v_{t\wedge\tau_n}|^2 dx \le C \Bigg[\mathbb{E} \int_{\mathscr{D}} |\nabla v_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}} \Big[|\nabla v_s|^2 + |f_s|^2 + |v_s|^2 + |\nabla g_s|_{\ell^2}^2 \Big] dx ds \Bigg].$$

Taking limit $n \to \infty$ and using Fatou's lemma, we get

$$\mathbb{E} \sup_{0 \le t \le T} \int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx \le C \left[\mathbb{E} \int_{\mathscr{D}} |\nabla v_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}} \left[|\nabla v_s|^2 + |f_s|^2 + |v_s|^2 + |\nabla g_s|_{\ell^2}^2 \right] dx ds \right]$$

where $C = C(K, d, \eta, \epsilon)$. Now note that the right hand side of above equation and (4.37) are independent of h and are finite and hence using e.g. [7, Ch. 5, Sec. 8, Theorem 3]), we get (4.27). $\square$

We now extend the result to the case $n = 2$ as follows. From Lemma 4.5 we have that v is a continuous $W^{1,2}(\mathscr{D}')$-valued adapted process such that $v \in L^2(\Omega \times (0,T); W^{2,2}(\mathscr{D}'))$, and it satisfies (4.26). If Assumptions A-4.5 and A-4.6 hold for $n = 2$, then from (4.26), we get

$$\begin{aligned} d(\partial_l v_t) &= \partial_l (L_t v_t + f_t) dt + \sum_{k \in \mathbb{N}} \partial_l (M_t^k v_t + g_t^k) dW_t^k \\ &= \big(L_t(\partial_l v_t) + \bar{f}_t\big) dt + \sum_{k \in \mathbb{N}} \big(M_t^k(\partial_l v_t) + \bar{g}_t^k\big) dW_t^k \end{aligned} \tag{4.39}$$

on $[0,T] \times \mathscr{D}'$, where

$$\bar{f}_t := \sum_{j=1}^d \partial_j \Big(\sum_{i=1}^d \partial_l a_t^{ij} \, \partial_i v_t \Big) + \sum_{i=1}^d \partial_l b_t^i \, \partial_i v_t + \partial_l c_t \, v_t + \partial_l f_t$$

and

$$\bar{g}_t^k := \sum_{i=1}^d \partial_l \sigma_t^{ik} \, \partial_i v_t + \partial_l \mu_t^k \, v_t + \partial_l g_t^k.$$

Using Assumptions A-4.5, A-4.6 with $n = 2$ we get,

$$\bar{f} \in L^2(\Omega \times (0,T); L^2(\mathscr{D}')) \quad \text{and} \quad \bar{g} \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}'; \ell^2)).$$

Thus replacing $f, g^k, \mathscr{D}$ in (4.26) by $\bar{f}, \bar{g}^k$ and $\mathscr{D}'$ respectively, we see that $z = \partial_l v$ satisfies (4.26). Clearly $z \in C([0,T]; L^2(\mathscr{D}'))$ almost surely and $z \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}'))$ and hence all the assumptions of Lemma 4.5 are satisfied for the new linear equation (4.39). Therefore for all open $\mathscr{D}'' \Subset \mathscr{D}'$, we have

$$\begin{aligned} &\mathbb{E} \sup_{0 \le t \le T} |\partial_i z_t|^2_{L^2(\mathscr{D}'')} + \mathbb{E} \int_0^T |\partial_i z_t|^2_{W^{1,2}(\mathscr{D}'')} dt \\ &\qquad \le C \left[\mathbb{E} \int_{\mathscr{D}'} |\nabla z_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}'} \left[|\nabla z_t|^2 + |\bar{f}_t|^2 + |z_t|^2 + |\nabla \bar{g}_t|^2_{\ell^2} \right] dx dt \right] \end{aligned}$$

which, substituting back the values of $\bar{f}, \bar{g}^k$ and $z = \partial_l v$ and then using Assumption A-4.5 with $n = 2$ and (4.27), gives

$$\begin{aligned} &\mathbb{E} \sup_{0 \le t \le T} |\partial_i \partial_l v_t|^2_{L^2(\mathscr{D}'')} + \mathbb{E} \int_0^T |\partial_i \partial_l v_t|^2_{W^{1,2}(\mathscr{D}'')} dt \\ &\le C \left[\mathbb{E} \int_{\mathscr{D}'} \sum_{|\gamma| \le 2} |D^\gamma v_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}'} \Big[\sum_{|\gamma| \le 2} |D^\gamma v_t|^2 + \sum_{|\gamma| \le 1} |D^\gamma f_t|^2 + \sum_{|\gamma| \le 2} |D^\gamma g_t|^2_{\ell^2} \Big] dx dt \right] \end{aligned}$$

for all $i = 1, \ldots, d$ and open $\mathscr{D}'' \Subset \mathscr{D}'$ where $C = C(\mathscr{D}'', d, T, K, \kappa)$. Repeating the above procedure k times, we have the following result.

Lemma 4.6. *Assume that v is a continuous $L^2(\mathscr{D})$-valued adapted process satisfying* (4.26) *and such that $v \in L^2(\Omega \times (0,T); W^{1,2}(\mathscr{D}))$. If Assumptions A-4.2, A-4.5 and A-4.6 hold for*

$n = k$, then

$$\mathbb{E}\sup_{0\le t\le T}|\partial_{i_k}\dots\partial_{i_1}v_t|^2_{L^2(\mathscr{D}^k)}+\mathbb{E}\int_0^T|\partial_{i_k}\dots\partial_{i_1}v_t|^2_{W^{1,2}(\mathscr{D}^k)}dt\le C\Bigg[\mathbb{E}\int_{\mathscr{D}^{k-1}}\sum_{|\gamma|\le k}|D^\gamma v_0|^2dx$$
$$+\mathbb{E}\int_0^T\int_{\mathscr{D}^{k-1}}\Big[\sum_{|\gamma|\le k}|D^\gamma v_t|^2+\sum_{|\gamma|\le k-1}|D^\gamma f_t|^2+\sum_{|\gamma|\le k}|D^\gamma g_t|^2_{\ell^2}\Big]dxdt\Bigg]$$

for all $i_k = 1,\dots,d$ and open $\mathscr{D}^k \Subset \mathscr{D}^{k-1}$ where $C = C(\mathscr{D}^k, d, T, K, \kappa)$.

We immediately see that Theorem 4.3 follows from Lemma 4.6. Using Theorems 4.1 and 4.3, we can now prove Theorem 4.2.

Proof of Theorem 4.2. Let u be the solution to (4.1) given by Theorem 4.1. Then considering $f_t(u_t,\nabla u_t)+f^0_t$ as a new free term f_t, we observe that u satisfies (4.26) with such free term. Now under the Assumptions A-4.3, A-4.4 and due to Theorem 4.1, applied with $p\ge 2\alpha-2$, we get the estimate (4.3) and hence

$$\begin{aligned}\mathbb{E}\int_0^T|f_t|^2_{L^2(\mathscr{D})}dt =&\mathbb{E}\int_0^T\int_{\mathscr{D}}|f(u_t,\nabla u_t)+f^0_t|^2dxdt\\ \le&2\Big[\mathbb{E}\int_0^T\int_{\mathscr{D}}K^2(1+|u_t|)^{2\alpha-2}dxdt+\mathbb{E}\int_0^T\int_{\mathscr{D}}|f^0_t|^2dxdt\Big]\\ \le&C\Big[1+\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}|u_t|^{2\alpha-2}dx\Big]+2\mathbb{E}\int_0^T\int_{\mathscr{D}}|f^0_t|^2dxdt<\infty.\end{aligned}$$

Hence we can apply Theorem 4.3 with $n = 1$ thus proving the first claim.

Moreover if f is a function of t,ω,x and r only such that (4.25) holds, then taking $f_t(u_t)+f^0_t$ as a new free term f_t, similarly as above, we get

$$\begin{aligned}\mathbb{E}\int_0^T|\partial_i f_t|^2_{L^2(\mathscr{D})}dt &= \mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i u_t\,\partial_r f_t(u_t)+\partial_i f_t(u_t)+\partial_i f^0_t|^2dxdt\\ &\le C\mathbb{E}\int_0^T\int_{\mathscr{D}}\big[|\nabla u_t|^2(1+|u_t|)^{2\alpha-4}+(1+|u_t|)^{2\alpha-2}+|\partial_i f^0_t|^2\big]dxdt\\ &\le C\mathbb{E}\int_0^T\int_{\mathscr{D}}\big[1+|\nabla u_t|^2+|\nabla u_t|^2|u_t|^{2\alpha-4}+|u_t|^{2\alpha-2}+|\partial_i f^0_t|^2\big]dxdt<\infty\end{aligned}$$

for any $i\in\{1,\dots,d\}$. Hence $f(u)+f^0\in L^2(\Omega\times(0,T);W^{1,2}(\mathscr{D}))$. Thus all the conditions of Theorem 4.3 are satisfied for $n = 2$. This yields the second claim. □

Remark 4.5. Note that to prove even higher regularity than that given by Theorem 4.2 one would need to show that

$$\mathbb{E}\int_0^T|\partial_j\partial_i f_t|^2_{L^2(\mathscr{D})}dt<\infty\,.$$

Using our approach we would require that

$$\mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_j u_t\partial_i u_t\partial_r^2 f_t(u_t)|^2\,dxdt<\infty\,.$$

However the L^p-estimates from Theorem 4.1 are not sufficient. To overcome this, one may try to formally apply ∂_i to the SPDE (4.1) and then to try to get the analogous L^p-estimates for the equation for the derivative. However, since the semilinear term is no longer monotone, the proof will break down.

4.3 Regularity in weighted spaces using L^p-theory & time regularity

In this section, we raise the regularity of the solution to the SPDE (4.1) using L^p-theory from Kim [20]. The reason for using L^p-theory is that one gets better estimates for the solution of the corresponding linear equation, see Theorem 4.4, given below, which follows immediately from [20, Theorem 2.9].

We will use this together with the L^p-estimates we proved in Theorem 4.1 to obtain regularity results (both space and time) for the solution of the semilinear equation (4.1), see Theorems 4.5 and 4.6 below. In particular we obtain Hölder continuity in time of order $\gamma < \frac{1}{2} - \frac{2}{q}$ for the solution to (4.1) as a process in weighted L^q-space, where q comes from the integrability assumptions imposed on the data.

First, we introduce some notations, concepts and assumptions from [20]. For $r_0 > 0$ and $x \in \mathbb{R}^d$, let $B_{r_0}(x) := \{y \in \mathbb{R}^d : |x - y| < r_0\}$.

Definition 4.2 (Domain of class C_u^1)**.** The domain $\mathscr{D} \subset \mathbb{R}^d$ is said to be of class C_u^1 if for any $x_0 \in \partial\mathscr{D}$, there exist $r_0, K_0, L_0 > 0$ and a one-one, onto continuously differentiable map $\Psi : B_{r_0}(x_0) \to G$, for a domain $G \subset \mathbb{R}^d$, satisfying the following:

(i) $\Psi(x_0) = 0$ and $\Psi\big(B_{r_0}(x_0) \cap \mathscr{D}\big) \subset \{y \in \mathbb{R}^d : y^1 > 0\}$,

(ii) $\Psi\big(B_{r_0}(x_0) \cap \partial\mathscr{D}\big) = G \cap \{y \in \mathbb{R}^d : y^1 = 0\}$,

(iii) $|\Psi|_{C^1(B_{r_0}(x_0))} \le K_0$ and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \le K_0|y_1 - y_2|$ for any $y_1, y_2 \in G$,

(iv) $|\Psi_x(x_1) - \Psi_x(x_2)| \le L_0|x_1 - x_2|$ for any $x_1, x_2 \in B_{r_0}(x_0)$.

Let $\mathscr{D}$ be of class C_u^1 and $\rho(x) := \operatorname{dist}(x, \partial\mathscr{D})$. Then, by [20, Lemma 2.5] and [23, Remark 2.7](since $\mathscr{D}$ is bounded), there exists a bounded real valued function ψ defined on $\bar{\mathscr{D}}$ satisfying

$$\sup_{x\in\mathscr{D}} \rho^{|\gamma|}(x)|D^\gamma \partial_i \psi(x)| < \infty \tag{4.40}$$

for any $i = 1, \dots, d$ and any multi-index γ, such that

$$\frac{1}{C}\rho \le \psi \le C\rho \ \text{ in } \ \mathscr{D},$$

for some constant C. In other words, ψ and ρ are comparable in $\mathscr{D}$, and in estimates they can be used interchangeably (up to multiplication by a constant). Moreover this implies $\psi \ge 0$.

For $1 \le q < \infty$, $\theta \in \mathbb{R}$ and a non-negative integer n, define the weighted Sobolev space $H_\theta^{n,q}(\mathscr{D})$ by

$$H_\theta^{n,q}(\mathscr{D}) := \{u : \rho^{|\gamma|+(\theta-d)/q} D^\gamma u \in L^q(\mathscr{D}) \ \text{ for any } \ |\gamma| \le n\},$$

where the norm for $u \in H_\theta^{n,q}(\mathscr{D})$ is given by

$$|u|^q_{H_\theta^{n,q}} := \sum_{i=0}^{n} \sum_{|\gamma|=i} \int_{\mathscr{D}} |D^\gamma u(x)|^q \rho^{\theta-d+iq}(x)dx.$$

For functions $u : \mathbb{R}^d \to \mathbb{R}^{d'}$, we define the norm analogously and use the same notation. The following result (see Lototsky [33, Theorem 4.1]) plays an important role in proving our results.

Remark 4.6. The following are equivalent:

(i) $u \in H_\theta^{n,q}(\mathscr{D})$,

(ii) $u \in H_\theta^{n-1,q}(\mathscr{D})$ and $\psi\partial_i u \in H_\theta^{n-1,q}(\mathscr{D})$ for all $i = 1, 2, \dots d$,

(iii) $u \in H_\theta^{n-1,q}(\mathscr{D})$ and $\partial_i(\psi u) \in H_\theta^{n-1,q}(\mathscr{D})$ for all $i = 1, 2, \dots d$.

Further, let

$$\mathbb{H}_\theta^{n,q}(\mathscr{D}) := L^q(\Omega \times (0,T), H_\theta^{n,q}(\mathscr{D})).$$

In the rest of the chapter, we assume that

$$q \geq 2 \quad \text{and} \quad d-2+q < \theta < d-1+q \tag{4.41}$$

so that in view of [20, Remark 2.7] and Assumption 4.7(v) below, the assumption regarding existence of an $\mathscr{A}_{p,\theta}$-type set (see Definition 2.6 and Assumption 2.8 in [20]), is satisfied. Finally, we need the following assumption on the coefficients:

A - 4.7. For any $i, j = 1, \dots, d$,

(i) the real valued coefficients a^{ij} and their spatial derivatives up to order $n+1$ are $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and bounded by K,

(ii) the real-valued coefficients b^i, c and their spatial derivatives up to order n are $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and are bounded by K,

(iii) the coefficients $\sigma^i = (\sigma^{ik})_{k=1}^\infty$, $\mu = (\mu^k)_{k=1}^\infty$ and their spatial derivatives up to order $n+1$ are ℓ^2-valued $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and almost surely

$$\sum_{i=1}^d \sum_{k\in\mathbb{N}} \sum_{|\gamma|\leq n+1} |D^\gamma \sigma_t^{ik}(x)|^2 + \sum_{k\in\mathbb{N}} \sum_{|\gamma|\leq n+1} |D^\gamma \mu_t^k(x)|^2 \leq K$$

for all t and x,

(iv) for almost every (t,ω), the coefficients $a^{ij}(t,x)$ and $\sigma^i(t,x)$ are uniformly continuous in $x \in \mathscr{D}$,

(v) and there exists constants $\delta_1, \delta_2 \in (0,1]$ such that

$$\delta_1|\xi|^2 \leq \delta_2 \sum_{i,j=1}^d a_t^{ij}(x)\xi_i\xi_j \leq \sum_{i,j=1}^d \Big(a_t^{ij}(x) - \frac{1}{2}\sum_{k\in\mathbb{N}} \sigma_t^{ik}(x)\sigma_t^{jk}(x)\Big)\xi_i\xi_j \leq \sum_{i,j=1}^d a_t^{ij}(x)\xi_i\xi_j \leq \frac{1}{\delta_1}|\xi|^2$$

for all $\omega \in \Omega$, $t \in [0,T]$, $x \in \mathscr{D}$, $\xi \in \mathbb{R}^d$.

Note that, the operator L given by (4.2) is in divergence form but the results from [20] are for operators in non-divergence form. One knows that (4.1) can be expressed in non-divergence form if the coefficients a^{ij} are differentiable. Thus Assumption A-4.7 implies Assumptions 2.2 and 2.3 in [20]. Hence the following theorem follows from Theorem 2.9 of Kim [20].

Theorem 4.4. *Assume $\mathscr{D}$ is of class C_u^1. Further, let Assumption A-4.7 hold with some $n \geq 0$. If $\psi f \in \mathbb{H}_\theta^{n,q}(\mathscr{D})$, $g \in \mathbb{H}_\theta^{n+1,q}(\mathscr{D};\ell^2)$ and $\psi^{\frac{2}{q}-1}\phi \in \mathbb{H}_\theta^{n+2,q}(\mathscr{D})$, then*

$$\begin{aligned} dv_t &= (L_t v_t + f_t)dt + \sum_{k\in\mathbb{N}} (M_t^k v_t + g_t^k)dW_t^k \quad \text{on } [0,T]\times\mathscr{D}, \\ v_t &= 0 \ \text{ on } \partial\mathscr{D}, \quad v_0 = \phi \ \text{ on } \mathscr{D}, \end{aligned} \tag{4.42}$$

has a unique solution v such that $\psi^{-1}v \in \mathbb{H}_\theta^{n+2,q}(\mathscr{D})$.

In fact Theorem 2.9 in Kim [20] is proved even for fractional weighted Sobolev spaces and under somewhat weaker assumptions. We do not use fractional spaces here to keep the presentation simpler. As to being able to use weaker assumptions: to obtain results for the semilinear equation (4.1) we will need to apply our results from Section 4.1, in particular Theorem 4.1 and thus we cannot substantially weaken our assumptions here. Finally, we can state the main results on regularity for the solution to semilinear SPDE (4.1).

Theorem 4.5. *Assume $\mathscr{D}$ is of class C_u^1 and u is the solution to (4.1). Further, let Assumptions A-4.3, A-4.4 with $p \geq \max(q\alpha - q, 2)$ and A-4.7 with $n = 0$ hold. If for some q satisfying (4.41), $\psi^{\frac{2}{q}-1}\phi \in \mathbb{H}_\theta^{2,q}(\mathscr{D})$, $g \in \mathbb{H}_\theta^{1,q}(\mathscr{D};\ell^2)$ and $f^0 \in \mathbb{H}_\theta^{0,q}(\mathscr{D})$, then $\psi^{-1}u \in \mathbb{H}_\theta^{2,q}(\mathscr{D})$.*

Moreover, in the case Assumption A-4.7 holds with $n = 1$ and almost surely,

$$\begin{aligned}
&|\partial_i f_t(x,r,z)| \le K(1+|r|)^{\alpha-1},\\
&|\partial_r f_t(x,r,z)| \le K(1+|r|)^{\alpha-2}\\
&\text{and } |\partial_z f_t(x,r,z)| \le K(1+|r|)^{\alpha-1}
\end{aligned}\tag{4.43}$$

for all $i = 1,\dots,d$, $t \in [0,T]$, $x \in \mathscr{D}$, $r \in \mathbb{R}$ and all $z \in \mathbb{R}^d$, if for some q satisfying (4.41), $\psi^{\frac{2}{q}-1}\phi \in \mathbb{H}^{3,q}_\theta(\mathscr{D})$, $g \in \mathbb{H}^{2,q}_\theta(\mathscr{D};\ell^2)$ and $f^0 \in \mathbb{H}^{1,q}_\theta(\mathscr{D})$, then $\psi^{-1}u \in \mathbb{H}^{3,\frac{q}{2}}_\theta(\mathscr{D})$.

Remark 4.7. Note that if $\psi^{-1}u \in \mathbb{H}^{2,q}_\theta(\mathscr{D})$, then by using Remark 4.6, we get

$$\psi^{-1}u \in \mathbb{H}^{1,q}_\theta(\mathscr{D}) \text{ and } \partial_i u \in \mathbb{H}^{1,q}_\theta(\mathscr{D}) \ \forall i = 1,2,\dots d.$$

Invoking Remark 4.6 again, we have

$$\psi^{-1}u \in \mathbb{H}^{0,q}_\theta(\mathscr{D}),\ \ \partial_i u \in \mathbb{H}^{0,q}_\theta(\mathscr{D}) \text{ and } \psi\partial_i\partial_j u \in \mathbb{H}^{0,q}_\theta(\mathscr{D}) \ \forall i,j = 1,2,\dots d. \tag{4.44}$$

Finally, we present the result on time regularity of the solution of (4.1).

Theorem 4.6. *Under the assumptions of Theorems 4.1 and 4.5,*

$$u \in C^\gamma\big([0,T]; H^{0,q}_{\theta+q}(\mathscr{D})\big) \quad a.s.$$

i.e., the solution u to SPDE (4.1), as a $H^{0,q}_{\theta+q}(\mathscr{D})$- valued process, is Hölder continuous of order γ for every $\gamma < \frac{1}{2} - \frac{2}{q}$ and q satisfying (4.41).

Note that one would like u to be Hölder continuous with exponent γ as a process with values in a weighted Sobolev space with the same weight exponent θ as in the results for spatial regularity (Theorem 4.5). However we need to use (4.44) in our arguments when proving Theorem 4.6 which leads to requiring the weight exponent to be $\theta + q$.

Before proving these theorems, we first prove the following lemma:

Lemma 4.7. *Let $\tilde\theta > d$ and $\tilde q \ge 1$. Further, let assumptions of Theorem 4.1 hold with $p \ge \max(\tilde q\alpha - \tilde q, 2)$ and $f^0 \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$. If u is the solution to (4.1) and $f_t := f_t(u_t,\nabla u_t) + f^0_t$, then $f \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$ and thus $\psi f \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$.*

Proof. First we note that $\tilde\theta > d$ and $\mathscr{D}$ is bounded, therefore $\sup_{x\in\mathscr{D}} \rho^{\tilde\theta-d}(x) < \infty$. Using this along with Assumption A-4.3 implies

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{\mathscr{D}} |f_t|^{\tilde q}\rho^{\tilde\theta-d}dxdt\\
&\quad= \mathbb{E}\int_0^T\int_{\mathscr{D}} |f_t(u_t,\nabla u_t) + f^0_t|^{\tilde q}\rho^{\tilde\theta-d}dxdt\\
&\quad\le C\Big[\mathbb{E}\int_0^T\int_{\mathscr{D}} (1+|u_t|)^{\tilde q\alpha-\tilde q}dxdt + \mathbb{E}\int_0^T\int_{\mathscr{D}} |f^0_t|^{\tilde q}\rho^{\tilde\theta-d}dxdt\Big]\\
&\quad\le C\Big[1 + \mathbb{E}\sup_{0\le t\le T}|u_t|^{\tilde q\alpha-\tilde q}_{L^{\tilde q\alpha-\tilde q}}\Big] + C\mathbb{E}\int_0^T\int_{\mathscr{D}} |f^0_t|^{\tilde q}\rho^{\tilde\theta-d}dxdt,
\end{aligned}\tag{4.45}$$

which is finite in view of Theorem 4.1 and the fact $f^0 \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$. Now note that ψ is bounded on $\bar{\mathscr{D}}$ and hence,

$$\mathbb{E}\int_0^T\int_{\mathscr{D}} |\psi f_t|^{\tilde q}\rho^{\tilde\theta-d}\,dxdt \le C\mathbb{E}\int_0^T\int_{\mathscr{D}} |f_t|^{\tilde q}\rho^{\tilde\theta-d}\,dxdt < \infty$$

as desired. ∎

Proof of Theorem 4.5. Let u be the solution to (4.1) given by Theorem 4.1. Then considering $f_t(u_t,\nabla u_t) + f^0_t$ as a new free term f_t, the solution u satisfies (4.42). We wish to apply

Theorem 4.4 with $n = 0$ and in order to do so we need to show that $\psi f \in \mathbb{H}^{0,q}_\theta(\mathscr{D})$. Indeed this follows immediately by using Lemma 4.7 with $\tilde\theta = \theta$ and $\tilde q = q$. Hence applying Theorem 4.4 with $n = 0$ we obtain $\psi^{-1}u \in \mathbb{H}^{2,q}_\theta(\mathscr{D})$. This completes the proof of the first statement of the theorem.

We now consider the case when Assumption A-4.7 holds with $n = 1$. Again we will apply Theorem 4.4 (but now with $n = 1$ and $\frac{q}{2}$ in place of q) and so we need to show that $\psi f \in \mathbb{H}^{1,\bar q}_\theta(\mathscr{D})$ with $\bar q := \frac{q}{2}$. Taking $\tilde\theta = \theta$ and $\tilde q = \bar q$ in Lemma 4.7, we get $\psi f \in \mathbb{H}^{0,\bar q}_\theta(\mathscr{D})$. Thus we consider

$$\mathbb{E}\int_0^T\int_{\mathscr{D}} |\partial_i(\psi f_t)|^{\bar q}\rho^{\theta-d+\bar q}dxdt = I_1 + I_2\,,$$

where,

$$I_1 := \mathbb{E}\int_0^T\int_{\mathscr{D}} |f_t|^{\bar q}|\partial_i\psi|^{\bar q}\rho^{\theta-d+\bar q}dxdt$$

and

$$I_2 := \mathbb{E}\int_0^T\int_{\mathscr{D}} |\partial_i f_t|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}dxdt\,.$$

Clearly $I_1 < \infty$ using (4.40), the fact ρ is bounded on $\mathscr{D}$ and Lemma 4.7 (with $\tilde\theta = \theta$ and $\tilde q = \bar q$). Further observe that,

$$\begin{aligned}\partial_i f_t &= \partial_i(f_t(u_t,\nabla u_t) + f^0_t)\\ &= \partial_i f_t(u_t,\nabla u_t) + \partial_i u_t\,\partial_r f_t(u_t,\nabla u_t) + \partial_i(\nabla u_t)\,\nabla_z f_t(u_t,\nabla u_t) + \partial_i f^0_t\end{aligned}$$

where $\nabla_z f_t$ is the gradient with respect to z of $f_t = f_t(x,r,z)$. Thus, we have

$$I_2 \le C(I_3 + I_4 + I_5 + I_6)$$

where,

$$\begin{aligned}I_3 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}} |\partial_i f_t(u_t,\nabla u_t)|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,,\\ I_4 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}} |\partial_i u_t\,\partial_r f_t(u_t,\nabla u_t)|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,,\\ I_5 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}} |\partial_i(\nabla u_t)\,\nabla_z f_t(u_t,\nabla u_t)|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\\ \text{and}\quad I_6 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}} |\partial_i f^0_t|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,.\end{aligned}$$

Now, using the fact that ψ and ρ are bounded on $\mathscr{D}$ and the assumption on growth of derivatives of the semilinear term, see (4.43), we observe that

$$\begin{aligned}I_3 &\le C\mathbb{E}\int_0^T\int_{\mathscr{D}} \big(1 + |\partial_i f_t(u_t,\nabla u_t)|\big)^q dxdt\\ &\le C\Big[1 + \mathbb{E}\int_0^T\int_{\mathscr{D}}(1+|u_t|)^{q\alpha-q}dxdt\Big]\,.\end{aligned}$$

This is finite in view of Theorem 4.1, see the estimate (4.45) for details. Further, using Young's inequality and the fact that ψ and ρ are bounded on $\mathscr{D}$ along with growth assumption (4.43), we get

$$\begin{aligned}I_4 &\le C\mathbb{E}\int_0^T\int_{\mathscr{D}} \Big[|\partial_i u_t|^q + |\partial_r f_t(u_t,\nabla u_t)|^q\Big]\rho^{\theta-d}dxdt\\ &\le C\Big[|\partial_i u|^q_{\mathbb{H}^{0,q}_\theta} + \mathbb{E}\int_0^T\int_{\mathscr{D}}(1+|u_t|)^{q\alpha-2q}dxdt\Big]\,.\end{aligned}$$

We see that this is finite using Remark 4.7 and Theorem 4.1 again. Furthermore, using Young's inequality, growth assumption (4.43) and the fact that ψ and ρ are comparable, we obtain

$$\begin{aligned} I_5 &\le C\mathbb{E}\int_0^T \int_{\mathscr{D}} \Big[|\partial_i(\nabla u_t)|^q + |\nabla_z f_t(u_t, \nabla u_t)|^q\Big]\psi^q \rho^{\theta-d} dxdt \\ &\le C\Big[|\psi\partial_i(\nabla u)|^q_{\mathbb{H}^{0,q}_\theta} + \mathbb{E}\int_0^T \int_{\mathscr{D}} (1+|u_t|)^{q\alpha-q} dxdt\Big]. \end{aligned}$$

Thus, applying Remark 4.7 and Theorem 4.1 as before, we obtain $I_5 < \infty$. Finally, the fact that ψ and ρ are comparable and bounded on $\mathscr{D}$ implies

$$I_6 \le C\mathbb{E}\int_0^T \int_{\mathscr{D}} \big(1+|\partial_i f^0_t|\big)^q \rho^{\theta-d+q} dxdt \le C\Big[1 + \mathbb{E}\int_0^T \int_{\mathscr{D}} |\partial_i f^0_t|^q \rho^{\theta-d+q} dxdt\Big]$$

which is finite since $f^0 \in \mathbb{H}^{1,q}_\theta(\mathscr{D})$. Thus $\psi f \in \mathbb{H}^{1,\bar{q}}_\theta(\mathscr{D})$ and we can apply Theorem 4.4 with $n = 1$ and $\bar{q}$ in place of q to complete the proof. $\square$

Proof of Theorem 4.6. We will prove the result using Kolmogorov continuity theorem. To ease the notation we let $f_t := f_t(u_t, \nabla u_t) + f^0_t$. Then from (4.1) we see that

$$\mathbb{E}|u_t - u_s|^q_{H^{0,q}_{\theta+q}} \le 2^{q-1}(I_1(s,t) + I_2(s,t)), \tag{4.46}$$

where,

$$I_1(s,t) := \mathbb{E}\Big|\int_s^t (L_r u_r + f_r) dr\Big|^q_{H^{0,q}_{\theta+q}} \text{ and } I_2(s,t) := \mathbb{E}\Big|\sum_{k\in\mathbb{N}} \int_s^t (M^k_r u_r + g^k_r) dW^k_r\Big|^q_{H^{0,q}_{\theta+q}}.$$

We note that $f^0 \in \mathbb{H}^{0,q}_\theta(\mathscr{D})$ implies $f^0 \in \mathbb{H}^{0,q}_{\theta+q}(\mathscr{D})$ because ρ is bounded on $\mathscr{D}$. Now using Hölder's inequality, we get

$$\begin{aligned} I_1(s,t) &\le (t-s)^{q-1}\mathbb{E}\int_s^t |L_r u_r + f_r|^q_{H^{0,q}_{\theta+q}} dr \\ &\le C(t-s)^{q-1}\Big[\mathbb{E}\int_s^t |L_r u_r|^q_{H^{0,q}_{\theta+q}} dr + \mathbb{E}\int_s^t |f_r|^q_{H^{0,q}_{\theta+q}} dr\Big]. \end{aligned} \tag{4.47}$$

Using Assumption A-4.7 with $n = 0$, we get

$$\begin{aligned} |L_r u_r|^q_{H^{0,q}_{\theta+q}} &= \int_{\mathscr{D}} \Big|\sum_{j=1}^d \partial_j\Big(\sum_{i=1}^d a^{ij}_t \partial_i u_r\Big) + \sum_{i=1}^d b^i_t \partial_i u_r + c_t u_r\Big|^q \rho^{\theta+q-d} dx \\ &\le C\int_{\mathscr{D}} \Big(\sum_{i,j=1}^d |\partial_i\partial_j u_r|^q + \sum_{i=1}^d |\partial_i u_r|^q + |u_r|^q\Big)\rho^{\theta+q-d} dx \\ &\le C\left(\sum_{i,j=1}^d |\psi\partial_i\partial_j u_r|^q_{H^{0,q}_\theta} + |\psi|^q_{C(\bar{\mathscr{D}})}\sum_{i=1}^d |\partial_i u_r|^q_{H^{0,q}_\theta} + |\psi|^{2q}_{C(\bar{\mathscr{D}})}|\psi^{-1}u_r|^q_{H^{0,q}_\theta}\right). \end{aligned}$$

Substituting this in (4.47) and using the fact that ψ is bounded on $\bar{\mathscr{D}}$, we obtain

$$\begin{aligned} I_1(s,t) &\le C(t-s)^{q-1}\left(\sum_{i,j=1}^d |\psi\partial_i\partial_j u|^q_{\mathbb{H}^{0,q}_\theta} + \sum_{i=1}^d |\partial_i u|^q_{\mathbb{H}^{0,q}_\theta} + |\psi^{-1}u|^q_{\mathbb{H}^{0,q}_\theta} + |f|^q_{\mathbb{H}^{0,q}_{\theta+q}}\right) \\ &\le C(t-s)^{q-1}, \end{aligned} \tag{4.48}$$

where last statement follows using Remark 4.7 and Lemma 4.7 with $\tilde{\theta} = \theta + q$ and $\tilde{q} = q$. Furthermore using Burkholder–Davis–Gundy's inequality, Assumption A-4.7 with $n = 0$, Hölder's inequality and the fact that ρ is bounded on $\mathscr{D}$, we see that

$$
\begin{aligned}
I_2(s,t) &= \mathbb{E}\int_{\mathscr{D}} \Big|\sum_{k\in\mathbb{N}}\int_s^t (M_r^k u_r + g_r^k)dW_r^k\Big|^q \rho^{\theta+q-d}dx \\
&\le \int_{\mathscr{D}} \mathbb{E}\Big[\int_s^t \sum_{k\in\mathbb{N}} |M_r^k u_r + g_r^k|^2 dr\Big]^{\frac{q}{2}} \rho^{\theta+q-d}dx \\
&= \int_{\mathscr{D}} \mathbb{E}\Big[\int_s^t \sum_{k\in\mathbb{N}} \Big|\sum_{i=1}^d \sigma_r^{ik}\partial_i u_r + \mu_r^k u_r + g_r^k\Big|^2 dr\Big]^{\frac{q}{2}} \rho^{\theta+q-d}dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_{\mathscr{D}} \mathbb{E}\Big[\int_s^t \Big(\sum_{i=1}^d |\partial_i u_r|^2 + |u_r|^2 + \sum_{k\in\mathbb{N}} |g_r^k|^2\Big)dr\Big]^{\frac{q}{2}} \rho^{\theta+q-d}dx \\
&\le C\int_{\mathscr{D}} (t-s)^{\frac{q}{2}-1}\mathbb{E}\Big[\int_s^t \Big(\sum_{i=1}^d |\partial_i u_r|^q + |u_r|^q + |g_r|_{\ell^2}^q\Big)dr\Big]\rho^{\theta+q-d}dx \qquad (4.49) \\
&\le C(t-s)^{\frac{q}{2}-1}\left(\sum_{i=1}^d |\partial_i u|^q_{\mathbb{H}^{0,q}_\theta} + |\psi^{-1}u|^q_{\mathbb{H}^{0,q}_\theta} + |g|^q_{\mathbb{H}^{0,q}_\theta}\right) \le C(t-s)^{\frac{q}{2}-1}\,.
\end{aligned}
$$

Here, the last inequality is obtained using Remark 4.7 as before and the assumption that $g\in\mathbb{H}^{1,q}_\theta(\mathscr{D};\ell^2)$. Using (4.48) and (4.49) in (4.46), we obtain

$$
\mathbb{E}|u_t - u_s|^q_{H^{0,q}_{\theta+q}} \le C|t-s|^{\frac{q}{2}-1}
$$

which on using Kolmogorov continuity theorem concludes the result. □

Corollary 4.1. Under the assumptions of Theorems 4.1, 4.2 (parts (i) and (ii)) and 4.5 we have,

$$
u\in C^\alpha\big([0,T];W^{1,2}(\mathscr{D}')\big) \quad a.s.
$$

for every $\alpha < \frac{1}{4}-\frac{1}{q}$ with q satisfying (4.41) and $\mathscr{D}'\Subset\mathscr{D}$.

Proof. Note that for any open $\mathscr{D}'\Subset\mathscr{D}$, there exists a constant $M>0$ such that the distance function ρ satisfies $|\rho(x)|\ge M$ for all $x\in\mathscr{D}'$. Therefore using Theorem 4.6, we get that almost surely

$$
\begin{aligned}
|u_t-u_s|_{L^q(\mathscr{D}')} &= \Big(\int_{\mathscr{D}'} |u_t-u_s|^q dx\Big)^{\frac{1}{q}} \le \Big(\sup_{x\in\mathscr{D}'}\frac{1}{\rho^{\theta+q-d}}\int_{\mathscr{D}} |u_t-u_s|^q\rho^{\theta+q-d}dx\Big)^{\frac{1}{q}} \\
&\le \frac{1}{(M^{\theta+q-d})^{\frac{1}{q}}}|t-s|^{\frac{1}{2}-\frac{2}{q}-\epsilon}|u|_{C^{\frac{1}{2}-\frac{2}{q}-\epsilon}\left([0,T];\,H^{0,q}_{\theta+q}(\mathscr{D})\right)}
\end{aligned}
$$

for any $\epsilon>0$ and all $s,t\in[0,T]$. Further, since $q\ge 2$, using Hölder's inequality we have that there exists a random variable C such that

$$
|u_t-u_s|_{L^2(\mathscr{D}')} \le C|t-s|^{\frac{1}{2}-\frac{2}{q}-\epsilon}
$$

which implies that almost surely $u\in C^{\frac{1}{2}-\frac{2}{q}-\epsilon}\big([0,T];L^2(\mathscr{D}')\big)$ for any $\epsilon>0$. Furthermore using Theorem 4.2, we have that almost surely $u\in C\big([0,T];W^{2,2}(\mathscr{D}')\big)$. Now using Gagliardo–Nirenberg inequality, we have that almost surely for all $s,t\in[0,T]$,

$$
\begin{aligned}
|u_t-u_s|_{W^{1,2}(\mathscr{D}')} &\le C|u_t-u_s|^{\frac{1}{2}}_{L^2(\mathscr{D}')}|u_t-u_s|^{\frac{1}{2}}_{W^{2,2}(\mathscr{D}')} \\
&\le C\Big(|t-s|^{\frac{1}{2}-\frac{2}{q}-\epsilon}|u|_{C^{\frac{1}{2}-\frac{2}{q}-\epsilon}\left([0,T];\,L^2(\mathscr{D}')\right)}\Big)^{\frac{1}{2}}\Big(2|u|_{C\left([0,T];\,W^{2,2}(\mathscr{D}')\right)}\Big)^{\frac{1}{2}} \\
&\le C|t-s|^{\frac{1}{4}-\frac{1}{q}-\frac{\epsilon}{2}}
\end{aligned}
$$

for some random variable C which concludes the result since $\epsilon>0$ is arbitrary. □

4.4 Application in numerical approximations

In this section, using the regularity results that we have obtained, we derive the rate of convergence of a space discretization scheme for a special case of (4.1).

We consider the semi-linear SPDE,

$$\begin{aligned} du_t &= (\Delta u_t - u_t^3)dt + u_t dW_t \quad \text{on } [0,T]\times(0,\pi) \\ u_t(0) &= u_t(\pi) = 0 \ \ \forall t \in [0,T], \\ u_0(x) &= \phi \ \ \forall x \in (0,\pi), \end{aligned} \tag{4.50}$$

where W is a one-dimensional Wiener process.

Note that (4.50) can be realised as SPDE (4.1) on domain $\mathscr{D} = (0,\pi) \subset \mathbb{R}$ taking coefficients $a = \mu = 1$, $b = c = \sigma = 0$, the semi-linear term $f(r) = -r^3$ and the free terms $f^0 = g = 0$. Clearly, Assumptions A-4.1 to A-4.4 are satisfied with $\alpha = 4$ and any $p \geq 4$ by assuming that $\phi \in L^p(\Omega, \mathscr{F}_0; L^p(0,\pi))$. Hence by Theorem 4.1, there exists a unique solution u to (4.50) in the sense of Definition 4.1 and

$$\mathbb{E}\sup_{0\leq t\leq T}|u_t|^p_{L^p((0,\pi))} + \mathbb{E}\int_0^T\int_0^\pi |Du_s|^2|u_s|^{p-2}dxds \leq C\mathbb{E}|\phi|^p_{L^p((0,\pi))}. \tag{4.51}$$

Further, Assumption A-4.5 holds for every $n \in \mathbb{N}$. Thus if $\phi \in L^2(\Omega; W^{1,2}(0,\pi))$, then by Theorem 4.2, for every compact $\mathscr{D}' \subset (0,\pi)$, we have

$$\mathbb{E}\int_0^T |u_t|^2_{W^{2,2}(\mathscr{D}')} < \infty.$$

We now want to define a space discretiztion scheme for (4.50). For that let $\{\phi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(0,\pi)$ consisting of the eigenvectors of the Laplace operator and for each $m \in \mathbb{N}$, we define an m-dimensional space V_m to be the space generated by $\{\phi_i : i = 1,2,\ldots m\}$. Further, let $\Pi_m : L^2((0,\pi)) \to V_m$ be the projection operators satisfying

$$|v - \Pi_m v|^2_{W_0^{1,2}((0,\pi))} \leq Cm^{-\delta}|v|^2_{V_\delta}, \quad \forall v \in W_0^{1,2}((0,\pi)), \tag{4.52}$$

for some $\delta > 0$ and a vector space V_δ such that $V_\delta \hookrightarrow W_0^{1,2}((0,\pi))$ and

$$\mathbb{E}\int_0^T |u|^2_{V_\delta} < \infty, \tag{4.53}$$

where u is the solution of (4.50).

For example, if we choose $\phi_i := \sqrt{\frac{2}{\pi}}\sin(ix)$, $i \in \mathbb{N}$, then $\{\phi_i\}_{i\in\mathbb{N}}$ forms an orthonormal basis of $L^2((0,\pi))$ s.t. $\Delta\phi_i = -i^2\phi_i$ for each $i \in \mathbb{N}$. In this case, (4.52) holds with $\delta = 2$ and $V_\delta = W^{2,2}((0,\pi))$. Indeed, for $v \in W_0^{1,2}((0,\pi))$

$$|v - \Pi_m v|^2_{W_0^{1,2}((0,\pi))} = |Dv - D(\Pi_m v)|^2_{L^2((0,\pi))} = \Big|\sum_{i=m+1}^\infty (v,\phi_i)D\phi_i\Big|^2_{L^2((0,\pi))} = \sum_{i=m+1}^\infty i^2|(v,\phi_i)|^2$$

where using integration by parts twice, we have

$$\begin{aligned} (v,\phi_i) &= \int_0^\pi \sqrt{\frac{2}{\pi}}\sin(ix)v(x)dx = \frac{1}{i}\int_0^\pi \sqrt{\frac{2}{\pi}}\cos(ix)Dv(x)dx \\ &= -\frac{1}{i^2}\int_0^\pi \sqrt{\frac{2}{\pi}}\sin(ix)D^2v(x)dx = -\frac{1}{i^2}(D^2v,\phi_i) \end{aligned}$$

and therefore,

$$|v-\Pi_m v|^2_{W_0^{1,2}((0,\pi))}=\sum_{i=m+1}^{\infty}\frac{1}{i^2}|(D^2v,\phi_i)|^2\leq\sum_{i=1}^{\infty}\frac{1}{m^2}|(D^2v,\phi_i)|^2\leq\frac{1}{m^2}|v|^2_{W^{2,2}((0,\pi))}.$$

Further, (4.53) is satisfied using the higher regularity results we have obtained in Theorem 4.2.

Define $u_0^m:=\Pi_m u_0$. Note that Π_m and Laplacian commute with each other since ϕ_i's are assumed to be the eigenvectors of the Laplace operator. Moreover using the Gagliardo–Nirenberg inequality and (4.53), we have

$$\begin{aligned}|v-\Pi_m v|^2_{L^4((0,\pi))}&\leq C|v-\Pi_m v|_{L^2((0,\pi))}|v-\Pi_m v|_{W_0^{1,2}((0,\pi))}\\&\leq C|v-\Pi_m v|^2_{W_0^{1,2}((0,\pi))}\leq Cm^{-\delta}|v|^2_{V_\delta}.\end{aligned}\tag{4.54}$$

For each $m\in\mathbb{N}$, we consider the following space-discretization scheme for the SPDE (4.50).

$$du_t^m=(\Pi_m\Delta u_t^m-\Pi_m(u_t^m)^3)dt+u_t^m dW_t,\quad t\in[0,T].\tag{4.55}$$

From well-known results (see, e.g., [28, Chapter 2, Theorem 3.1]), SDE (4.55) has a unique strong solution.

4.4.1 Rate of convergence for the space discretization scheme

Now we derive the rate of convergence for the space discretization scheme (4.55).

Theorem 4.7. *Under the assumptions* (4.52) *and* (4.53)*, we have*

$$\mathbb{E}|\Pi_m u_t-u_t^m|^2_{L^2}\leq Cm^{-\frac{2\delta}{3}}.$$

Proof. From (4.50), we have

$$\Pi_m u_t=\Pi_m u_0+\int_0^t(\Pi_m\Delta u_s-\Pi_m(u_s)^3)ds+\Pi_m u_s dW_s\,.$$

Further, the numerical scheme (4.55) can be rewritten as,

$$u_t^m=u_0^m+\int_0^t(\Pi_m\Delta u_s^m-\Pi_m(u_s^m)^3)ds+\int_0^t u_s^m dW_s$$

Therefore, the error term

$$\begin{aligned}e_t^m:=&\Pi_m u_t-u_t^m\\=&\Pi_m u_0-u_0^m+\int_0^t\Big(\Pi_m\Delta u_s-\Pi_m\Delta u_s^m-\Pi_m(u_s)^3+\Pi_m(u_s^m)^3\Big)ds+\int_0^t\big(\Pi_m u_s-u_s^m\big)dW_s\end{aligned}$$

for all $t\in[0,T]$. By the application of Itô's formula, we obtain

$$\begin{aligned}|\Pi_m u_t-u_t^m|^2_{L^2}=&|\Pi_m u_0-u_0^m|^2_{L^2}+2\int_0^t\Big\langle\Pi_m u_s-u_s^m,\Pi_m\Delta u_s-\Pi_m\Delta u_s^m\Big\rangle ds\\&-2\int_0^t\Big\langle\Pi_m u_s-u_s^m,\Pi_m(u_s)^3-\Pi_m(u_s^m)^3\Big\rangle ds\\&+\int_0^t|\Pi_m u_s-u_s^m|^2_{L^2}ds+2\int_0^t(\Pi_m u_s-u_s^m,\Pi_m u_s-u_s^m)dW_s\end{aligned}$$

which on taking expectation gives,

$$\mathbb{E}|\Pi_m u_t-u_t^m|^2_{L^2}=\mathbb{E}|\Pi_m u_0-u_0^m|^2_{L^2}+E_1+E_2+\mathbb{E}\int_0^t|\Pi_m u_s-u_s^m|^2_{L^2}ds\tag{4.56}$$

where,

$$E_1 := 2\mathbb{E}\int_0^t \big\langle \Pi_m u_s - u_s^m, \Pi_m \Delta u_s - \Pi_m \Delta u_s^m \big\rangle ds$$

and

$$E_2 := -2\mathbb{E}\int_0^t \big\langle \Pi_m u_s - u_s^m, \Pi_m (u_s)^3 - \Pi_m (u_s^m)^3 \big\rangle ds.$$

Using the fact that Π_m and Δ commute and applying integration by parts, we get the following estimates for the term E_1.

$$\begin{aligned}
E_1 &= 2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, \Pi_m \Delta u_s - \Pi_m \Delta u_s^m \big\rangle ds \\
&= 2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, \Delta \Pi_m u_s - \Delta u_s^m \big\rangle ds = -2\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^2_{W_0^{1,2}} ds
\end{aligned}$$

Further using the property of projection operator Π_m and the inequality,

$$\big(|x|^{\alpha-2}x - |y|^{\alpha-2}y\big)\,(x-y) \geq c_\alpha |x-y|^\alpha \quad \forall x, y \in \mathbb{R}^d,\ \alpha \geq 2$$

with $\alpha = 4$, there exists a constant $c > 0$ such that

$$\begin{aligned}
E_2 =& -2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, \Pi_m (u_s)^3 - \Pi_m (u_s^m)^3 \Big\rangle ds \\
=& -2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, (u_s)^3 - (u_s^m)^3 \Big\rangle ds \\
=& -2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, (u_s)^3 - (\Pi_m u_s)^3 \Big\rangle ds - 2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, (\Pi_m u_s)^3 - (u_s^m)^3 \Big\rangle ds \\
\leq& -2\mathbb{E}\int_0^t \Big\langle \Pi_m u_s - u_s^m, (u_s)^3 - (\Pi_m u_s)^3 \Big\rangle ds - c\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds.
\end{aligned}$$

Applying Hölder's inequality in the first term and observing that $|a^3 - b^3| \leq |a-b|(1+|a|+|b|)^2$ for all $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
E_2 \leq& C\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|_{L^4} |(u_s)^3 - (\Pi_m u_s)^3\big|_{L^{\frac{4}{3}}} ds - c\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds \\
\leq& C\mathbb{E}\int_0^t \big|\Pi_m u_s - u_s^m\big|_{L^4} \big|u_s - \Pi_m u_s\big|_{L^4} \big|1 + |u_s| + |\Pi_m u_s|\big|^2_{L^4} ds - c\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds
\end{aligned}$$

and then application of Young's inequality yields,

$$E_2 \leq C\mathbb{E}\int_0^t \big|u_s - \Pi_m u_s\big|^{\frac{4}{3}}_{L^4} \big|1 + |u_s| + |\Pi_m u_s|\big|^{\frac{8}{3}}_{L^4} - (c-\epsilon)\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds.$$

Again applying Hölder's inequality and using (4.54), the term E_2 can be estimated as follows.

$$\begin{aligned}
E_2 \leq& C\Big[\mathbb{E}\int_0^t \big|u_s - \Pi_m u_s\big|^2_{L^4} ds\Big]^{\frac{2}{3}} \Big[\mathbb{E}\int_0^t \big|1 + |u_s| + |\Pi_m u_s|\big|^8_{L^4} ds\Big]^{\frac{1}{3}} \\
&- (c-\epsilon)\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds \\
\leq& C\Big[m^{-\delta}\mathbb{E}\int_0^T |u_s|^2_{V_\delta} ds\Big]^{\frac{2}{3}} \Big[\mathbb{E}\int_0^t \big|1 + |u_s| + |\Pi_m u_s|\big|^8_{L^8} ds\Big]^{\frac{1}{3}} \\
&- (c-\epsilon)\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds \\
\leq& -(c-\epsilon)\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|^4_{L_4} ds + Cm^{\frac{-2\delta}{3}}
\end{aligned}$$

where the last inequality has been obtained using (4.51) and (4.53). On combining the estimates of E_1 and E_2, (4.56) gives

$$\mathbb{E}|\Pi_m u_t - u_t^m|_{L^2}^2 + 2\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|_{W_0^{1,2}}^2 ds + (c-\epsilon)\mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|_{L^4}^4 ds$$
$$= \mathbb{E}|\Pi_m u_0 - u_0^m|_{L^2}^2 + Cm^{\frac{-2\delta}{3}} + \mathbb{E}\int_0^t |\Pi_m u_s - u_s^m|_{L^2}^2 ds$$

and applying Gronwall's lemma, we get

$$\mathbb{E}|\Pi_m u_t - u_t^m|_{L^2}^2 \leq e^T\big(\mathbb{E}|\Pi_m u_0 - u_0^m|_{L^2}^2 + Cm^{\frac{-2\delta}{3}}\big),$$

which on using $u_0^m = \Pi_m u_0$ concludes the result. □

In this toy example we have seen that the raising the regularity twice (even once in this case) is enough to find the rate of convergence of the proposed space discretization scheme. This is part of the ongoing research work, where we are trying to find the rate of convergence of space-time discretization schemes for semilinear SPDEs of type (4.1).

Chapter 5

Future work

In Chapter 4, we have obtained higher regularity results for semilinear SPDEs (4.1) with Dirichlet boundary condition. As we have observed in Example 1.1 that even for a very good data, one can not have have a good solution upto the boundary. So, following Krylov's approach to deal with difficulties that arise due to boundaries, we used the distance function $\rho(x) := \operatorname{dist}(x, \mathscr{D})$ to define the weighted Sobolev spaces in Chapter 4 where, the weights allow the spatial derivatives to oscillate or explode near the boundary. We note that in Example 1.1, the contradiction arises when we apply the Dirichlet boundary condition. A natural question to ask is - do we face similar problem if we have Neumann boundary condition instead of Dirichlet boundary condition. If yes, can we use same approach to deal with the problem? If yes, will the same weight function work? In future, we are aiming to answer such questions.

Here one should note that if we manage to raise the regularity once, then the problem of raising the regularity further will reduce to raising regularity for a Dirichlet boundary value problem since the Neumann boundary condition for the solution u becomes the Dirichlet boundary condition for the derivative Du of the solution. Thus, we aim to raise the regularity of SPDEs with Neumann boundary condition only once.

We now present the partial results that we have obtained so far in this direction.

Let $T > 0$ be given, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis, $\mathscr{P}$ be the predictable σ-algebra and $W := (W_t)_{t\in[0,T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. Further, let $\mathbb{R}^d_+ := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0\}$ be the d-dimensional half-space. For integers $n \geq 0$, $p \geq 2$, we use the notation $\mathbb{H}^{n,p} := L^p(\Omega \times (0,T); W^{n,p}(\mathbb{R}^d))$ and $\mathbb{H}^{n,p}_+ := L^p(\Omega \times (0,T); W^{n,p}(\mathbb{R}^d_+))$.

Consider the following Neumann boundary value problem on the half space:

$$\begin{aligned} du_t &= \Big(\sum_{i=1}^d a^{ii}_t D_{ii} u_t + \sum_{i=1}^d D_i f^i_t + f^0_t\Big)dt + \sum_{k\in\mathbb{N}} (\mu^k_t u_t + g^k_t) dW^k_t \quad \text{on } [0,T]\times\mathbb{R}^d_+ \\ u_0 &= 0 \text{ on } \mathbb{R}^d_+, \quad \frac{\partial u_t}{\partial \eta} = 0 \text{ on } \partial\mathbb{R}^d_+, \end{aligned} \tag{5.1}$$

where $\eta(x)$ is the unit exterior normal vector at $x \in \mathbb{R}^d_+$.

We make the following assumptions:

A - 5.1. There exists a constant $\kappa > 0$ such that $a^{ii} \geq \kappa$ for all $t \in [0,T]$, $x \in \mathbb{R}^d_+$, $\omega \in \Omega$ and $i = 1, 2, \ldots, d$.

A - 5.2. For any $i = 1, 2, \ldots, d$, the real valued coefficients a^{ii} are $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d_+)$-measurable and bounded by K. Further, the coefficient $\mu = (\mu^k)_{k\in\mathbb{N}}$ is ℓ^2-valued $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d_+)$-measurable and almost surely,

$$\sum_{k\in\mathbb{N}} |\mu^k_t(x)|^2 \leq K$$

for all t and x.

A - 5.3. For any $i = 1, 2, \ldots, d$, a^{ii} is continuous in x, uniformly in (t, ω) and μ is Lipschitz continuous in x.

A - 5.4. The forcing terms $D_i f^i, f^0 \in \mathbb{H}^{0,p}_+$ and $g^k \in \mathbb{H}^{1,p}_+$.

We now define the following even extensions on the whole space.

$$\tilde{a}^{ii}_t(x) := \begin{cases} a^{ii}_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ a^{ii}_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0, \end{cases}$$

$$\tilde{f}^0_t(x) := \begin{cases} f^0_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ f^0_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0, \end{cases}$$

$$\tilde{\mu}^k_t(x) := \begin{cases} \mu^k_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ \mu^k_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0, \end{cases}$$

$$\tilde{g}^k_t(x) := \begin{cases} g^k_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ g^k_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0, \end{cases}$$

$$\tilde{f}^i_t(x) := \begin{cases} f^i_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ f^i_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0, \end{cases}$$

for $i = 2, 3, \ldots, d$. Further, we define the odd extension,

$$\tilde{f}^1_t(x) := \begin{cases} f^1_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ -f^1_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0 \end{cases}$$

and consider the following extension of (5.1) on the whole space, i.e.

$$\begin{aligned} d\bar{u}_t &= \Big(\sum_{i=1}^d \tilde{a}^{ii}_t D_{ii} \bar{u}_t + \sum_{i=1}^d D_i \tilde{f}^i_t + \tilde{f}^0_t \Big) dt + \sum_{k \in \mathbb{N}} (\tilde{\mu}^k_t \bar{u}_t + \tilde{g}^k_t) dW^k_t \quad \text{on} \ [0, T] \times \mathbb{R}^d, \\ \bar{u}_0 &= 0 \ \text{ on } \ \mathbb{R}^d. \end{aligned} \tag{5.2}$$

Note that in view of above definitions and Assumptions A-5.1 to A-5.4, the cofficients $\tilde{a}$ and $\tilde{\mu}$ satisfy Assumptions A-5.1 to A-5.3 on the whole space. Further, $D_i \tilde{f}^i, \tilde{f}^0 \in \mathbb{H}^{0,p}$ and $\tilde{g}^k \in \mathbb{H}^{1,p}$. Therefore using results from L^p-theory of SPDEs (see, e.g. Krylov [25]), there exists a unique solution $\bar{u} \in \mathbb{H}^{2,p}$ solving (5.2), i.e. for all $\phi \in C^\infty_0(\mathbb{R}^d)$,

$$\begin{aligned} (\bar{u}_t, \phi) = \int_0^t \Big\{ \sum_{i=1}^d \Big(-\tilde{a}^{ii}_s D_i \bar{u}_s, D_i \phi \Big) + \Big(-\sum_{i=1}^d D_i \tilde{a}^{ii}_s + \sum_{i=1}^d D_i \tilde{f}^i_s + \tilde{f}^0_s, \phi \Big) \Big\} ds \\ + \sum_{k \in \mathbb{N}} \int_0^t (\tilde{\mu}^k_s \bar{u}_s + \tilde{g}^k_s, \phi) dW^k_s \end{aligned}$$

and $\bar{u}_0 = 0$ on $\mathbb{R}^d$. Working with $\phi \in C^\infty_0(\mathbb{R}^d)$ such that $\phi(x_1, \ldots, x_d) = 0$ for $(x_1, \ldots, x_d)$ with $x_1 < 0$, we get that $u := \bar{u}|_{\mathbb{R}^d_+} \in \mathbb{H}^{2,p}_+$ satisfies (5.1) such that $u_0 = 0$ on $\mathbb{R}^d_+$. However, it remains to show that u satisfies the Neumann boundary condition in (5.1). Indeed, by defining

$$\tilde{u}_t(x) := \begin{cases} u_t(x_1, \ldots, x_d) & \text{if} \quad x_1 \geq 0 \\ u_t(-x_1, \ldots, x_d) & \text{if} \quad x_1 < 0, \end{cases}$$

and then substituting $(-x_1, x_2, \ldots, x_d)$ in (5.1) if $x_1 < 0$, we get

$$\begin{aligned} d\tilde{u}_t &= \Big(\sum_{i=1}^d \tilde{a}^{ii}_t D_{ii} \tilde{u}_t + \sum_{i=1}^d D_i \tilde{f}^i_t + \tilde{f}^0_t \Big) dt + \sum_{k \in \mathbb{N}} (\tilde{\mu}^k_t u_t + \tilde{g}^k_t) dW^k_t \quad \text{on} \ [0, T] \times \mathbb{R}^d, \\ \tilde{u}_0 &= 0 \ \text{ on } \ \mathbb{R}^d, \end{aligned}$$

which means that $\tilde{u} \in \mathbb{H}^{2,p}$ is also a L^p-solution of (5.2). Thus by uniqueness both $\bar{u}$ and $\tilde{u}$ are same which means that $\bar{u}(x_1, x_2, \ldots, x_d) = \bar{u}(-x_1, x_2, \ldots, x_d)$. Since $D_1 \bar{u}$ is continuous, we must have $D_1 u(0, x_2, \ldots, x_d) = D_1 \bar{u}(0, x_2, \ldots, x_d) = 0$ meaning that u satisfies the Neumann boundary condition in (5.1) and hence is a solution of (5.1).

Thus, we observe that in this particular Neumann Boundary value problem, we could raise the regularity of the solution without requiring any weights. However, this method does not work in general as shown in the remark below.

Remark 5.1. The method discussed above fails to work if we allow for the term involving gradient of the solution under the stochastic integral. Indeed, consider the following Neumann boundary value problem:

$$\begin{aligned} du_t &= D^2 u_t dt + \sigma_t D u_t dW_t \quad \text{on } [0,T] \times \mathbb{R}_+ \,, \\ u_0 &= 0 \ \text{ on } \mathbb{R}_+, \quad \frac{\partial u_t}{\partial \eta} = 0 \ \text{ on } \partial \mathbb{R}_+, \end{aligned}$$

where $\eta(x)$ is the unit exterior normal vector at $x \in \mathbb{R}_+$.

In order to apply the above argument, one would require an odd extension of σ, viz.

$$\tilde{\sigma}_t(x) := \begin{cases} \sigma_t(x) & \text{if} \quad x \geq 0 \\ -\sigma_t(-x) & \text{if} \quad x < 0. \end{cases}$$

Since we want $\tilde{\sigma}$ to be Lipschitz continuous in x, we must have $\sigma_t(0) = 0$ for all $t \in [0,T]$.

One thing to note here is that to find a regular solution of the SPDE (5.1), we worked with even extentions of the free term g and the solution u unlike the odd continuations used by Krylov to solve the analogous problem with Dirichlet boundary condition while developing the theory.

Appendix A

Hilbert-space valued Wiener process

Many authors consider stochastic evolution equations with respect to cylindrical Q-Wiener process $\mathcal{W}$ taking values in a separable Hilbert space $(U, (\cdot,\cdot)_U, |\cdot|_U)$. Here Q is a linear, symmetric, non-negative definite and bounded operator on U. For an overview of stochastic integrals with respect to Hilbert-space valued Wiener processes, one may refer to Dalang and Sardanyons [5] or [40]. The operator under the stochastic integral would be taking values in the space of Hilbert–Schmidt operators, denoted by $L_2(U,H)$. The stochastic evolution equation considered is then written as

$$u_t = u_0 + \int_0^t A_s(u_s)ds + \int_0^t B_s(u_s)d\mathcal{W}_s, \quad t \in [0,T], \tag{A.1}$$

instead of (1.1) or (2.1). In this appendix, we show that these formulations are equivalent.

First we show that the stochastic Itô integral with respect to cylindrical Q-Wiener process on a separable Hilbert space can be expressed in the form of infinite sum of stochastic Itô integrals with respect to independent one-dimensional Wiener processes as considered in (1.1) or (2.1). Here $\mathcal{W}$ is cylindrical Q-Wiener process in U with $Q = I$, the identity on U. Let $(u^j)_{j\in\mathbb{N}}$ be an orthonormal basis of U, which in this case are also the eigenvectors of Q corresponding to the eigenvalues $(\lambda^j)_{j\in\mathbb{N}}$ where $\lambda^j = 1$ for each $j \in \mathbb{N}$.

For $t \in [0,T]$ and $j \in \mathbb{N}$, define $W_t^j := (\mathcal{W}_t, u^j)_U$. Then it can be seen that the processes $(W_t^j)_{t\in[0,T]}, j \in \mathbb{N}$ are independent real-valued Wiener processes. However, the series $\sum_{j=1}^\infty W_t^j u^j = \sum_{j=1}^\infty \sqrt{\lambda^j} W_t^j u^j$ does not converge in $L^2(\Omega;U)$ as $\sum_{j=1}^\infty \lambda^j$, i.e. trace of Q, is not finite. Consider the linear map $J: U \to U$ given by,

$$Ju := \sum_{j=1}^\infty \frac{1}{j}(u,u^j)_U u^j \quad \forall u \in U.$$

Note that $Ju^j = \frac{1}{j}u^j$ for each $j \in \mathbb{N}$. It can then be seen that J is an injective mapping satisfying,

$$\sum_{j=1}^\infty |Ju^j|_U^2 < \infty$$

and for each $t \in [0,T]$, the series

$$\sum_{j=1}^\infty (\mathcal{W}_t, u^j)_U Ju^j = \sum_{j=1}^\infty W_t^j Ju^j$$

converges in $L^2(\Omega;U)$. In fact, the series converges in $L^2(\Omega;C([0,T];U))$ and defines a Q_1-Wiener process on U where $Q_1 := JJ^*$ is a bounded linear operator on U which is nonnegative definite, symmetric and has finite trace. Moreover, $Q_1^{\frac{1}{2}}(U) = J(U)$ and $(Ju^j)_{j\in\mathbb{N}}$ forms an orthonormal basis of $J(U)$ where the norm on the space $Q_1^{\frac{1}{2}}(U) = J(U)$ is given by,

$$|Ju|_{Q_1^{\frac{1}{2}}(U)} = |Q_1^{-\frac{1}{2}} Ju|_U = |u|_U \quad \forall u \in U.$$

For details, one may refer to [40, Proposition 2.5.2].

Next we show that the two formulations of stochastic integral (with respect to cylindrical Q-Wiener process, or written as an infinite sum) are equivalent. Consider a progressively measurable process $(B_t)_{t\in[0,T]}$ taking values in $L_2(U;H)$, where $L_2(U;H)$ is the space of Hilbert Schmidt operators from U to H. Note that

$$B_t(\omega) \in L_2(U;H) \iff B_t(\omega) \circ J^{-1} \in L_2(J(U);H) = L_2(Q_1^{\frac{1}{2}}(U);H)$$

and then the stochastic integral with respect to cylindrical Q-Wiener processes is defined by the following:

$$\int_0^t B_s \, d\mathcal{W}_s := \int_0^t B_s \circ J^{-1} \, d\mathcal{W}_s \,, \quad t \in [0,T] \,,$$

where the integral on right-hand-side is with respect to Q_1-Wiener process on U (see, e.g., [40, Section 2.5.2]).

Now we show that the above stochastic integral with respect to a cylindrical Wiener process can be expressed as an infinite sum of stochastic integrals of suitable H-valued processes with respect to independent real-valued Wiener processes. Define $B_t^j := B_t(u^j) = (B_t \circ J^{-1})(Ju^j)$ for all $t \in [0,T]$ and $j \in \mathbb{N}$. Then $(B_t^j)_{j\in\mathbb{N}} \in \ell^2(H)$ since $B_t \in L_2(U;H)$. Further for $u \in U$, we have

$$B_t(u) = (B_t \circ J^{-1})(Ju) = \sum_{j=1}^{\infty} (u,u^j)_U (B_t \circ J^{-1})(Ju^j) = \sum_{j=1}^{\infty} (u,u^j)_U B_t^j$$

and hence

$$\int_0^t B_s d\mathcal{W}_s = \int_0^t B_s \circ J^{-1} d\mathcal{W}_s = \sum_{j=1}^{\infty} \int_0^t B_s^j \, dW_s^j \,. \tag{A.2}$$

Thus (A.2) implies that u is a solution to (2.1) if and only if it is a solution to (A.1).

Moreover assumptions in Chapter 2 made on operators $B^j : [0,T] \times \Omega \times V \to H$ can be equivalently replaced by assumptions on the operator $B : [0,T] \times \Omega \times V \to L_2(U;H)$ as follows. Assumption A-2.2 can be equivalently replaced by:

$\tilde{\text{A}}$ - 2 (Local Monotonicity)**.** Almost surely for all $t \in [0,T]$ and $x, \bar{x} \in V$,

$$2\langle A_t(x) - A_t(\bar{x}), x - \bar{x}\rangle + |B_t(x) - B_t(\bar{x})|^2_{L_2(U,H)} \le L(1 + |\bar{x}|_V^\alpha)(1 + |\bar{x}|_H^\beta)|x - \bar{x}|_H^2$$

and A-2.3 can be equivalently replaced by:

$\tilde{\text{A}}$ - 3 (p_0-Stochastic Coercivity)**.** Almost surely for all $t \in [0,T]$ and $x \in V$,

$$2\langle A_t(x), x\rangle + (p_0 - 1)|B_t(x)|^2_{L_2(U,H)} + \theta |x|_V^\alpha \le f_t + K|x|_H^2.$$

For the same reason, SEE (3.3) can be equivalently written as

$$du_t = \sum_{i=1}^{k} A_t^i(u_t)dt + B_t(u_t)d\mathcal{W}_t + \int_{\mathcal{D}^c} \gamma_t(u_t,z)\tilde{N}(dt,dz) + \int_{\mathcal{D}} \gamma_t(u_t,z)N(dt,dz) \tag{A.3}$$

and assumptions in Chapter 3 made on operators $B^j : [0,T] \times \Omega \times V \to H$ can be equivalently replaced by assumptions on the operator $B : [0,T] \times \Omega \times V \to L_2(U;H)$ as follows.

Assumption A-3.2 can be equivalently replaced by:

$\tilde{\mathbf{A}}$ **- 4** (Local Monotonicity)**.** Almost surely for all $t \in [0,T]$ and $x, \bar{x} \in V$,

$$2\sum_{i=1}^{k}\langle A_t^i(x) - A_t^i(\bar{x}), x - \bar{x}\rangle_i + |B_t(x) - B_t(\bar{x})|^2_{L_2(U,H)} + \int_{\mathcal{D}^c} |\gamma_t(x,z) - \gamma_t(\bar{x},z)|^2_H \nu(dz)$$
$$\leq \Big[L' + L''\Big(1 + \sum_{i=1}^{k} |\bar{x}|^{\alpha_i}_{V_i}\Big)(1 + |\bar{x}|^{\beta}_H)\Big]|x - \bar{x}|^2_H \,.$$

Finally A-3.3 can be equivalently replaced by:

$\tilde{\mathbf{A}}$ **- 5** (p_0-Stochastic Coercivity)**.** Almost surely for all $t \in [0,T]$ and $x \in V$,

$$2\sum_{i=1}^{k}\langle A_t^i(x), x\rangle_i + (p_0 - 1)|B_t(x)|^2_{L_2(U,H)} + \theta\sum_{i=1}^{k}|x|^{\alpha_i}_{V_i} + \int_{\mathcal{D}^c} |\gamma_t(x,z)|^2_H \nu(dz) \leq f_t + K|x|^2_H.$$

Bibliography

[1] D. Breit, *Regularity theory for nonlinear systems of SPDEs* , arXiv:1311.2001v4 (2016).

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[3] Z. Brzeźniak, W. Liu and J. Zhu, *Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise*, Nonlinear Anal. Real World Appl. (2014) 17, 283–310.

[4] Z. Brzeźniak and M. Veraar, *Is the stochastic parabolicity condition dependent on p and q ?*, Electron. J. Probab. (2012)17, 1–24.

[5] R. C. Dalang, L. Q. Sardanyons, *Stochastic integrals for spde's: A comparison*, Expo. Math. (2011) 29, 67–109.

[6] K. Dareiotis, M. Gerencsér, *On the boundedness of solutions of SPDEs*, Stoch. Partial Differ. Equ. Anal. Comput. (2015) 3(1), 84–102.

[7] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, 1998.

[8] F. Flandoli, *Dirichlet boundary value problem for stochastic parabolic equations: compatibility relations and regularity of solutions*, Stochastics (1990) 29(3), 331–357.

[9] M. Gerencsér, *Boundary regularity of stochastic PDEs*, Ann. Probab. (2019) 47(2), 804–834.

[10] B. Gess, *Strong solutions for stochastic partial differential equations of gradient type*, J. Funct. Anal. (2012) 263(8), 2355–2383.

[11] I. Gyöngy, *On stochastic equations with respect to semimartingales III*, Stochastics (1982) 7, 231–254.

[12] I. Gyöngy, N. V. Krylov, *On stochastic equations with respect to semimartingales I*, Stochastics (1980) 4, 1–21.

[13] I. Gyöngy, N. V. Krylov, *On stochastic equations with respect to semimartingales II. Ito formula in Banach spaces*, Stochastics (1982) 6, 153–173.

[14] I. Gyöngy, N. V. Krylov, *On the Rate of Convergence of Splitting-up Approximations for SPDEs*, Progr. Probab. (2003) 56, 301–321.

[15] I. Gyöngy, A. Millet, *Rate of Convergence of Space Time Approximations for Stochastic Evolution Equations*, Potential Anal. (2009) 30, 29–64.

[16] I. Gyöngy, D.Šiška, *Itô Formula for Processes Taking Values in Intersection of Finitely Many Banach Spaces*, Stoch. Partial Differ. Equ. Anal. Comput. (2017) 5(3), 428–455.

[17] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, North-Holland and Kodansha, Tokyo, 1989.

[18] A. Jentzen, M. Röckner, *Regularity analysis for stochastic partial differential equations with nonlinear multiplicative trace class noise*, J. Differential Equations (2012) 252(1), 114–136.

[19] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Second Edition, Springer, New York, 1991.

[20] K. -H. Kim, *On stochastic partial differential equations with variable coefficients in C^1 domains*, Stochastic Process. Appl. (2004) 112, 261–283.

[21] I. Kim, K. -H. Kim, *Some L_p and Hölder Estimates for Divergence Type Nonlinear SPDEs on C^1-Domains*, Potential Anal. (2014) 41, 583–612.

[22] I. Kim, K. -H. Kim, *A Regularity Theory for Quasi-Linear Stochastic Partial Differential Equations in Weighted Sobolev Spaces*, Stochastic Process. Appl. (2018) 128(2), 622–643.

[23] K. -H. Kim, N. V. Krylov, *On the Sobolev space theory of parabolic and elliptic equations in C^1 domains*, SIAM J. Math. Anal. (2004) 36(2), 618–642.

[24] N. V. Krylov, *A W_2^n-theory of the Dirichlet problem for SPDEs in general smooth domains*, Probab. Theory Related Fields (1994) 98, 389–421.

[25] N. V. Krylov, *On L_p-theory of stochastic partial differential equations in the whole space*, Siam J. Math. Anal. (1996) 27(2), 313–340.

[26] N. V. Krylov, *An analytic approach to SPDEs, Stochastic partial differential equations: six perspectives*, Math. Surveys Monogr. 64, Amer. Math. Soc., Providence, 1999.

[27] N. V. Krylov, *A relatively short proof of Itô's formula for SPDEs and its applications*, Stoch. Partial Differ. Equ. Anal. Comput. (2013) 1(1), 152–174.

[28] N. V. Krylov, B. L. Rozovskii, *Stochastic Evolution Equations*, Journal of Soviet Mathematics (1981) 14, 1233–1277.

[29] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier-Villars, Paris, 1969.

[30] W. Liu, M. Röckner, *SPDE in Hilbert Space with Locally Monotone Coefficients*, J. Funct. Anal. (2010) 259, 2902–2922.

[31] W. Liu, M. Röckner, *Local and global well-posedness of SPDE with generalized coercivity conditions*, J. Differential Equations. (2013) 254(2), 725–755.

[32] W. Liu, M. Röckner, *Stochastic Partial Differential Equations: An Introduction*, Springer, Cham, 2015.

[33] S. V. Lototsky, *Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations*, Methods Appl. Anal. (2000) 7(1), 195–204.

[34] X. Mao, *Stochastic Differential Equations and Applications*, Second Edition, Horwood Publishing, Chichester, 2007.

[35] R. Mikulevicius, H. Pragarauskas, *On L_p-estimates of some singular integrals related to jump processes*, SIAM J. Math. Anal. (2012) 44(4), 2305–2328.

[36] Neelima, D. Šiška, *Coercivity condition for higher order moments for nonlinear SPDEs and existence of solution under local monotonicity*, arXiv:1610.05700 (2016).

[37] Neelima, D. Šiška, *L^p-estimates and regularity for SPDEs with monotone semilinearity*, arxiv:1705.10232 (2017).

[38] Neelima, *On Well-Posedness of Stochastic anisotropic p-Laplace Equation driven by Lévy noise*, arXiv:1811.07855 (2018).

[39] E. Pardoux, *Equations aux dérivées partielles stochastiques non lineaires monotones. Etude des solutions forte de type Ito*, Thése Doct. Sci. Math. Univ. Paris Sud., 1975.

[40] C. Prévot, M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes in Mathematics, 1905, Springer, Berlin, 2007.

[41] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, 1st edition, Springer, Berlin, 1991.

[42] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, Berlin, 2005.

[43] B.L. Rozovskii, *Stochastic Evolution Systems. Linear Theory and Applications to Nonlinear Filtering*, Kluwer, Dordrecht, 1990.